

On the Integral Geometry of Liouville Billiard Tables

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Abstract

The notion of a Radon transform is introduced for completely integrable billiard tables. In the case of Liouville billiard tables of dimension 3 we prove that the Radon transform is one-to-one on the space of continuous functions K on the boundary which are invariant with respect to the corresponding group of symmetries. We prove also that the frequency map associated with a class of Liouville billiard tables is non-degenerate. This allows us to obtain spectral rigidity of the corresponding Laplace-Beltrami operator with Robin boundary conditions.

1 Introduction

This paper is concerned with the integral geometry and the spectral rigidity of Liouville billiard tables. By a billiard table we mean a smooth compact connected Riemannian manifold (X, g) of dimension $n \geq 2$ with a non-empty boundary $\Gamma := \partial X$. The elastic reflection of geodesics at Γ determines continuous curves on X called *billiard trajectories* as well as a discontinuous dynamical system on T^*X – the “*billiard flow*” – that generalizes the geodesic flow on closed manifolds without boundary. The billiard flow on T^*X induces a discrete dynamical system in the open coball bundle $B^*\Gamma$ of Γ given by the corresponding billiard ball map B and its iterates. The map B is defined in an open subset of $B^*\Gamma = \{\xi \in T^*\Gamma : \|\xi\|_g < 1\}$, where $\|\xi\|_g$ denotes the norm induced by the Riemannian metric g on the corresponding cotangent plane and it can be considered as a discrete Lagrangian systems as in [9], [11], [15]. The orbits of B can be obtained by a variational principal and they can be viewed as “discrete geodesics” of the corresponding Lagrangian. In this context, periodic orbits of B can be considered as “discrete closed geodesics”.

Let μ be a positive continuous function on $B^*\Gamma$. Denote by π_Γ^*K the pull-back of the continuous function $K \in C(\Gamma)$ with respect to the projection $\pi_\Gamma : T^*\Gamma \rightarrow \Gamma$. We are interested in the following problems.

Problem A. Let K be a continuous function on Γ such that the mean value of the product $\pi_\Gamma^*K \cdot \mu$ is zero on any periodic orbit of the billiard ball map B . Does it imply $K \equiv 0$?

The mapping assigning to any periodic orbit $\gamma = \{\varrho_0, \varrho_1, \dots, \varrho_{m-1}\} \subset B^*\Gamma$ of the map B the mean value $(1/m) \sum_{j=0}^{m-1} (\pi_\Gamma^*K \cdot \mu)(\varrho_j)$ of the function $\pi_\Gamma^*K \cdot \mu$ on γ can be viewed as a discrete analogue of the Radon transform, considering the periodic orbits of the billiard ball map as discrete closed geodesics. Problem A has a positive answer for any ball in the Euclidean space \mathbb{R}^n centered at the origin if $\mu = 1$ and K is even. In fact, approximating the great circles on the sphere by closed billiard trajectories of the billiard table we obtain from the hypothesis in Problem A that the integral of K over any great circle is zero. Since K is even, by Funk’s

theorem we obtain $K \equiv 0$ ([3, Theorem 4.53]). The case of general Riemannian manifold is much more complicated.

Denote by $\pi_X : T^*X \rightarrow X$ the natural projection of the cotangent bundle T^*X onto X . Let $S^*X|_\Gamma = \{\xi \in T^*X : \pi_X(\xi) \in \Gamma, \|\xi\|_g = 1\}$ be the restriction of the unit co-sphere bundle to Γ . There are two natural choices for the function μ we are concerned with, namely, $\mu \equiv 1$ or $\mu(\xi) = \langle \pi^+(\xi), n_g \rangle^{-1}$, $\xi \in B^*\Gamma$, where $\langle \cdot, \cdot \rangle$ is the standard pairing between vectors and covectors, n_g is the inward unit normal to Γ at $x = \pi_\Gamma(\xi)$, and $\pi^+ : B^*\Gamma \rightarrow S^*X|_\Gamma$ assigns to any $\xi \in T_x^*\Gamma$ with norm $\|\xi\|_g < 1$ the unit outgoing covector the restriction of which to $T_x\Gamma$ coincides with ξ . Recall that a covector based on x is outgoing if its value on $n_g(x)$ is non-negative. The latter choice of μ is related with the wave-trace formula for manifolds with boundary obtained by V. Guillemin and R. Melrose [4, 5]. It appears also in the iso-spectral invariants of the Robin boundary problem for the Laplace-Beltrami operator obtained in [12]. From now on we fix the positive function $\mu \in C(B^*\Gamma)$ by

$$\mu \equiv 1, \quad \text{or by} \quad \mu(\xi) = \langle \pi^+(\xi), n_g \rangle^{-1}, \quad \xi \in B^*\Gamma. \quad (1.1)$$

For that choice of μ , it will be shown that Problem A has a positive solution for a class of Liouville billiard tables of classical type. A Liouville billiard table (shortly L.B.T.) of dimension $n \geq 2$, is a completely integrable billiard table (X, g) (the notion of complete integrability will be recalled in Sect. 2) admitting n functionally independent and Poisson commuting integrals of the billiard flow on T^*X which are quadratic forms in the momentum. A L.B.T. can be viewed as a 2^{n-1} -folded branched covering of a disk-like domain in \mathbb{R}^n by the cylinder $\mathbb{T}^{n-1} \times [-N, N]$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $N > 0$. Liouville billiard tables of dimension two are defined in [10] and in any dimension $n \geq 2$ in [11], where the integrability of the billiard ball map is shown via the geodesic equivalence principal. Here we write explicitly first integrals of the billiard flow and show that it is completely integrable (see Sect. 3.1). An important subclass of L.B.T.s are the Liouville billiard tables of classical type having an additional symmetry and for which the boundary is strictly geodesically convex (with respect to the outward normal $-n_g$). It turns out that the group of isometries of a L.B.T. of classical type is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Moreover, the group of isometries of (X, g) induces a group of isometries G on Γ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. An important example of a L.B.T. of classical type is the interior of the n -axial ellipsoid equipped with the Euclidean metric. More generally, there is a non-trivial two-parameter family of L.B.T.s of classical type of constant scalar curvature κ having the same broken geodesics (considered as non-parameterized curves) as the ellipsoid [11, Theorem 3]. This family includes the ellipsoid ($\kappa = 0$), a L.B.T. on the sphere ($\kappa = 1$) and a L.B.T. in the hyperbolic space ($\kappa = -1$).

Theorem 1. *Let (X, g) , $\dim X = 3$, be an analytic L.B.T. of classical type. Suppose that there is at least one non-periodic geodesic on the boundary Γ . Choose μ as in (1.1). Let $K \in C(\Gamma)$ be invariant with respect to the group of isometries $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ of the boundary Γ and such that the mean value of $\pi_\Gamma^* K \cdot \mu$ on any periodic orbit of the billiard ball map is zero. Then $K \equiv 0$.*

In particular, Problem A has a positive solution for ellipsoidal billiard tables in \mathbb{R}^3 with $\mu \equiv 1$ as well as for $\mu(\xi) = \langle \pi^+(\xi), n_g \rangle^{-1}$, for any $K \in C(\Gamma)$ which is invariant under the reflections with respect to the coordinate planes O_{xy} , O_{yz} , and O_{xz} . More generally, Theorem 3 can be applied for any L.B.T. of the family described in [11, Theorem 3]. The condition that the boundary contains at least one non-closed geodesic will become clear after the discussion of Problem C.

As it was mentioned above the map assigning to each periodic orbit of the billiard ball map B the mean value of $\pi_\Gamma^* K \cdot \mu$ on it can be considered as a discrete analogue of the Radon transform. Another version of the Radon transform can be defined as follows. Denote by \mathcal{F} the family of all Lagrangian tori $\Lambda \subset B^*\Gamma$ which are invariant with respect to some exponent B^m , $m \geq 1$, of the billiard ball map B , i.e. $B^m(\Lambda) \subseteq \Lambda$. For any continuous function K on Γ we denote by $\mathcal{R}_{K,\mu}(\Lambda)$ the mean value of the integral of $\pi_\Gamma^* K \cdot \mu$ on $\Lambda \in \mathcal{F}$ with respect to the Leray form (see Sect. 2). The mapping $\Lambda \mapsto \mathcal{R}_{K,\mu}(\Lambda)$, $\Lambda \in \mathcal{F}$, will be called a Radon transform of K as well.

Problem B. Let K be a continuous function on Γ which is invariant with respect to the group of isometries G . Does the relation $\mathcal{R}_{K,\mu} \equiv 0$ imply $K \equiv 0$?

The main result of the paper is the following theorem, which gives a positive answer of Problem B for L.B.T.s.

Theorem 2. *Let (X, g) , $\dim X = 3$, be a Liouville billiard table of classical type. Fix μ by (1.1). If $K \in C(\Gamma)$ is invariant under the group of symmetries G of Γ and $\mathcal{R}_{K,\mu}(\Lambda) = 0$ for any $\Lambda \in \mathcal{F}$, then $K \equiv 0$.*

We point out that L.B.T.s of classical type are smooth by construction but they are not supposed to be analytic.

A similar result has been obtained for the ellipse in [4] and more generally for L.B.T.s of classical type in dimension $n = 2$ in [10] and [12]. It is always interesting to find a smaller set of data Λ for which the Radon transform is one-to-one. In the case $n = 2$ the proof is done by analyticity, and we need to know the values of the Radon transform $\mathcal{R}_{K,\mu}(\Lambda)$ only on a family of invariant circles $\{\Lambda_j\}_{j \in \mathbb{N}}$ approaching the boundary $S^*\Gamma$ of $B^*\Gamma$. The case $n = 3$ is more complicated, since the argument using analyticity does not work any more. Nevertheless, we can restrict the Radon transform to data “close” to the boundary in the following sense: It will be shown in Sect. 3.3 that any L.B.T. of classical type of dimension 3 admits four not necessarily connected charts U_j , $1 \leq j \leq 4$, of action-angle variables in $B^*\Gamma$. Two of them, say U_1 and U_2 , have the property that any unparameterized geodesic in $S^*\Gamma$ can be obtained as a limit of orbits of B lying either in U_1 or in U_2 (then the corresponding broken geodesics approximate geodesics of the boundary). Moreover, in any connected component of U_1 and U_2 there is such a sequence of orbits of B , while U_3 and U_4 do not enjoy this property. In other words, the charts U_1 and U_2 can be characterized by the property that there is a family of “*whispering gallery rays*” issuing from any of their connected components. For this reason the two cases $j = 1, 2$ will be referred as to boundary cases. Denote by \mathcal{F}_b the set of all $\Lambda \in \mathcal{F}$ lying either in U_1 or in U_2 . We will show in Theorem 4.1 that the restriction of the Radon transform $\mathcal{R}_{K,\mu}$ on \mathcal{F}_b determines uniquely K .

As an application we prove spectral rigidity of the Robin boundary problem for Liouville billiard tables. Given a real-valued function $K \in C(\Gamma, \mathbb{R})$, we consider the “positive” Laplace-Beltrami operator Δ on X with domain

$$D := \left\{ u \in H^2(X) : \frac{\partial u}{\partial n_g}|_\Gamma = Ku|_\Gamma \right\},$$

where $H^2(X)$ is the Sobolev space, and $n_g(x)$, $x \in \Gamma$, is the inward unit normal to Γ with respect to the metric g . We denote this operator by $\Delta_{g,K}$. It is a selfadjoint operator in $L^2(X)$ with discrete spectrum

$$\text{Spec } \Delta_{g,K} := \{\lambda_1 \leq \lambda_2 \leq \dots\},$$

where each eigenvalue $\lambda = \lambda_j$ is repeated according to its multiplicity, and it solves the spectral problem

$$\begin{cases} \Delta u = \lambda u & \text{in } X, \\ \frac{\partial u}{\partial n_g}|_{\Gamma} = Ku|_{\Gamma}. \end{cases} \quad (1.2)$$

Let $[0, 1] \ni t \mapsto K_t \in C^\infty(\Gamma, \mathbb{R})$ be a continuous family of smooth real-valued functions on Γ . To simplify the notations we denote by Δ_t the corresponding operators Δ_{g, K_t} . This family is said to be isospectral if

$$\forall t \in [0, 1], \operatorname{Spec}(\Delta_t) = \operatorname{Spec}(\Delta_0). \quad (1.3)$$

We consider here a weaker notion of isospectrality which has been introduced in [12]. Fix two positive constants c and $d > 1/2$, and consider the union of infinitely many disjoint intervals

(H₁) $\mathcal{I} := \cup_{k=1}^{\infty} [a_k, b_k], \quad 0 < a_1 < b_1 < \dots < a_k < b_k < \dots, \quad \text{such that}$

$$\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} b_k = +\infty, \quad \lim_{k \rightarrow \infty} (b_k - a_k) = 0, \quad \text{and } a_{k+1} - b_k \geq cb_k^{-d} \text{ for any } k \geq 1.$$

We impose the following “weak isospectral assumption”:

(H₂) *There is $a > 0$ such that $\forall t \in [0, 1], \operatorname{Spec}(\Delta_t) \cap [a, +\infty) \subset \mathcal{I}$, where \mathcal{I} is given by (H₁)*.

Using the asymptotics of the eigenvalues λ_j as $j \rightarrow \infty$ we have shown in [12] that the condition (H₁)-(H₂) is “natural” for any $d > n/2$ ($n = \dim X$), which means that the usual isospectral assumption (1.3) implies (H₁)-(H₂) for any such d and any $c > 0$.

Theorem 3. *Let (X, g) be a 3-dimensional analytic Liouville billiard table of classical type such that the boundary Γ has at least one non-periodic geodesic. Let*

$$[0, 1] \ni t \mapsto K_t \in C^\infty(\Gamma, \mathbb{R})$$

be a continuous family of real-valued functions on Γ satisfying the isospectral condition (H₁)-(H₂). Suppose that K_0 and K_1 are invariant with respect to the group of symmetries $G = (\mathbb{Z}/2\mathbb{Z})^3$ of Γ . Then $K_0 \equiv K_1$.

A similar result has been proved in [12] for smooth 2-dimensional billiard tables. The idea of the proof of Theorem 3 is as follows. Fix the continuous function μ by $\mu(\xi) = \langle \pi^+(\xi), n_g \rangle^{-1}$. First, using [12, Theorem 1.1] we obtain that

$$\mathcal{R}_{K_1, \mu}(\Lambda) = \mathcal{R}_{K_0, \mu}(\Lambda), \quad (1.4)$$

for any Liouville torus Λ of a frequency vector satisfying a suitable Diophantine condition. Next, we prove that the union of such tori is dense in the union of the two charts U_j , $j = 1, 2$, of “action-angle” coordinates in $B^*\Gamma$, which implies (1.4) for any torus $\Lambda \in \mathcal{F}_b$. Now the claim follows from Theorem 4.1. In the same way we prove Theorem 1. First we obtain that $\mathcal{R}_{K, \mu}(\Lambda) = 0$ for a set of “rational tori” Λ . Then we prove that the union of these tori is dense in $U_1 \cup U_2$, and we apply Theorem 4.1. We point out that the proof of Theorem 3 presented in Sect. 6 requires only finite smoothness of K_t (see Theorem 6.1).

An important ingredient in the proof of both theorems is the density of the corresponding families of invariant tori in U_j , $j = 1, 2$. This follows from the non-degeneracy of the frequency

map for Liouville billiard tables of classical type studied in Sect. 5. Recall that in any chart U_j of action-angles coordinates the frequency map assigns to any value of the momentum map the frequency vector of the minimal power $B^m : U_j \rightarrow U_j$, $m \geq 1$, that leaves invariant the corresponding Liouville tori $\Lambda \subset U_j$. The frequency map is said to be non-degenerate in U_j if its Hessian with respect to the action variables is non-degenerate in a dense subset of U_j . We are interested in the following problem:

Problem C. Is the frequency map non-degenerate in any chart of action-angle coordinates?

We prove in Theorem 5.1 that this is true in the charts U_j , $j = 1, 2$, for any *analytic L.B.T. of classical type for which the boundary Γ admits at least one non-closed geodesic*. The 3-axial ellipsoid and more generally any billiard table of the two-parameter family of L.B.T.s of classical type of constant scalar curvature described in [11, Theorem 3] has these properties.

The non-degeneracy of the frequency map appears also as a hypothesis in the Kolmogorov-Arnold-Moser theorem. In particular, Theorem 5.1 allows us to apply the KAM theorem for the billiard ball maps associated with small perturbations of the L.B.T.s in [11, Theorem 3]. It is a difficult problem to prove that the frequency map of a specific completely integrable system is non-degenerate. The non-degeneracy of the frequency map of completely integrable Hamiltonian systems has been systematically investigated in [7]. The main idea in [7] is to investigate the system at the singularities of the momentum map. In our case we reduce the system at the boundary $S^*\Gamma$ of $B^*\Gamma$. To our best knowledge this problem has not been rigorously studied for completely integrable billiard tables even in the case of the billiard table associated with the interior of the ellipsoid.

The article is organized as follows. In Sect. 2 we recall certain facts about the billiard ball map and define a Radon transform for completely integrable billiard tables. Sect. 3 is concerned with the construction of L.B.T.s. First we consider a cylinder $C = \mathbb{T}_{\omega_1} \times \mathbb{T}_{\omega_2} \times [-N, N]$, where $\mathbb{T}_l = \mathbb{R}/l\mathbb{Z}$ for $l > 0$ and $N > 0$ and define a “metric” g and two Poisson commuting quadratic with respect to the impulses integrals I_1 and I_2 of g in C . The non-negative quadratic form g is *degenerate* at a submanifold S of C . To make g a Riemannian metric we consider its push-forward on the quotient $\sigma : C \rightarrow \tilde{C}$ of C with respect to the group generated by two commuting involutions σ_1 and σ_2 whose fix point set is just S . The main result in this section is Proposition 3.3 which provides \tilde{C} with a differentiable structure such that the push-forwards $\tilde{g} := \sigma_* g$, $\tilde{I}_1 := \sigma_* I_1$ and $\tilde{I}_2 := \sigma_* I_2$ are smooth forms, \tilde{g} is a Riemannian metric on \tilde{C} and \tilde{I}_1 and \tilde{I}_2 are Poisson commuting integrals of \tilde{g} . In Sect. 3.3 we write an explicit parameterization of the regular tori by means of the values of the momentum map corresponding to the integrals \tilde{I}_1 and \tilde{I}_2 . The injectivity of the Radon transform is investigated in Sect. 4. The non-degeneracy of the frequency map of an analytic L.B.T. is investigated in Sect. 5. The proof of Theorem 1 and Theorem 3 is given in Sect. 6. In the Appendix we investigate the frequency map and the action-angle coordinates of completely integrable billiard tables and derive a formula for the frequency vectors of B^m .

2 Invariant manifolds, Leray form, and Radon transform

In the present section we define the Radon transform for integrable billiard tables. First we recall the definition of the billiard ball map B associated to a billiard table (X, g) , $\dim X = n$, with boundary Γ . Denote by $H \in C^\infty(T^*X, \mathbb{R})$ the Hamiltonian corresponding to the Riemannian

metric g on X via the Legendre transformation and set

$$S^*X := \{\xi \in T^*X : H(\xi) = 1\}, \quad S^*X|_\Gamma := \{\xi \in S^*X : \pi_X(\xi) \in \Gamma\},$$

$$S_\pm^*X|_\Gamma := \{\xi \in S^*X|_\Gamma : \pm\langle \xi, n_g \rangle > 0\},$$

n_g being the inward unit normal to Γ . Denote by $r : T^*X|_\Gamma \rightarrow T^*X|_\Gamma$ the “reflection” at the boundary given by $r : v \mapsto w$, where $w|_{T_y\Gamma} = v|_{T_y\Gamma}$ and $\langle w, n_g \rangle + \langle v, n_g \rangle = 0$. Obviously $r : S^*X|_\Gamma \rightarrow S^*X|_\Gamma$. Take $u \in S_+^*X|_\Gamma \subset T^*X$ and consider the integral curve $\gamma(t; u)$ of the Hamiltonian vector field X_H on T^*X starting at u . If it intersects transversally $S^*X|_\Gamma$ at a time $t_1 > 0$ and lies entirely in the interior of S^*X for $t \in (0, t_1)$, we set $\mathcal{B}_0(u) := \gamma(t_1, u) \in S_-^*X|_\Gamma$. The set $\mathcal{O} \subseteq S_+^*X|_\Gamma$ of all such u is open in $S_+^*X|_\Gamma$. The billiard ball map is defined by

$$\mathcal{B} := r \circ \mathcal{B}_0 : \mathcal{O} \rightarrow S_-^*X|_\Gamma.$$

Denote by $B^*\Gamma := \{\xi \in T^*\Gamma : H(\xi) < 1\}$ the (open) coball bundle of Γ . The natural projection $\pi_+ : S_+^*X|_\Gamma \rightarrow B^*\Gamma$ assigning to each $u \in S^*X|_\Gamma$ the covector $u|_{T_x\Gamma} \in B^*\Gamma$ admits a smooth inverse map $\pi^+ : B^*\Gamma \rightarrow S_+^*X|_\Gamma$. The map $B := \pi_+ \circ \mathcal{B} \circ \pi^+$ is defined in the open subset $\pi_+(\mathcal{O})$ of the coball bundle of Γ and it is a smooth symplectic map, i.e. it preserves the canonical symplectic two-form $\omega = dp \wedge dx$ on $B^*\Gamma$. The map B will be called a *billiard ball map* as well.

From now on we assume that the billiard ball map $B : B^*\Gamma \rightarrow B^*\Gamma$ is globally defined and *completely integrable*. By definition¹, the complete integrability of the billiard ball map of (X, g) means that there exist $n - 1$ invariant with respect to B smooth functions F_1, \dots, F_{n-1} on $B^*\Gamma$ which are functionally independent and in involution with respect to the canonical Poisson bracket on $T^*\Gamma$, i.e.

$$\{F_i, F_j\} = 0, \quad 1 \leq i, j \leq n - 1.$$

The functions F_1, \dots, F_{n-1} are said to be functionally independent in $B^*\Gamma$ if the form $dF_1 \wedge \dots \wedge dF_{n-1}$ does not vanish almost everywhere. A function f on $B^*\Gamma$ is said to be invariant with respect to the billiard ball map B if $B^*f = f$. The invariant functions with respect to the billiard ball map are called also *integrals*. In particular, as F_1, \dots, F_{n-1} are integrals, then any non-empty level set

$$L_c := \{\xi \in B^*\Gamma : F_1(\xi) = c_1, \dots, F_{n-1}(\xi) = c_{n-1}\}, \quad c = (c_1, \dots, c_{n-1}) \in \mathbb{R}^{n-1},$$

is invariant with respect to the billiard ball map $B : B^*\Gamma \rightarrow B^*\Gamma$. By Arnold-Liouville theorem any regular compact component Λ_c of L_c is diffeomorphic to the $(n - 1)$ -dimensional torus \mathbb{T}^{n-1} and there exists a tubular neighborhood of Λ_c in $B^*\Gamma$ symplectically diffeomorphic to $\mathbb{D}_r^{n-1} \times \mathbb{T}^{n-1}$ that is supplied with the canonical symplectic structure $\sum_{k=1}^{n-1} dJ_k \wedge d\theta_k$. Here $\mathbb{D}_r^{n-1} := \{J = (J_1, \dots, J_{n-1}) \in \mathbb{R}^{n-1} : |J| < r\}$ for some $r > 0$, $\theta = (\theta_1, \dots, \theta_{n-1})$ are the periodic coordinates on \mathbb{T}^{n-1} , and $|\cdot|$ is the Euclidean norm in \mathbb{R}^{n-1} . The coordinates (J, θ) are called *action-angle* coordinates of the billiard ball map. Recall that Λ_c is *regular* if the $(n - 1)$ -form $dF_1 \wedge \dots \wedge dF_{n-1}$ does not vanish at the points of Λ_c . Any regular torus Λ_c is a Lagrangian submanifold of $B^*\Gamma$ and it is also called a *Liouville torus*.

Assume that the Liouville torus Λ_c is invariant with respect to B^m for some $m \geq 1$, i.e. $B^m(\Lambda_c) = \Lambda_c$. Let α_c be a $(n - 1)$ -form defined in a tubular neighborhood of Λ_c in $B^*\Gamma$ so that

$$\omega^{n-1} := \omega \wedge \dots \wedge \omega = \alpha_c \wedge dF_1 \wedge \dots \wedge dF_{n-1}. \tag{2.1}$$

¹This is one of the many definitions of complete integrability of billiard ball map.

It follows from (2.1) that the restriction $\lambda_c := \alpha_c|_{\Lambda_c}$ of α_c to Λ_c is uniquely defined. The form λ_c is a volume form on Λ_c which is called *Leray form*. As B preserves both the symplectic structure ω and the functions F_1, \dots, F_{n-1} , one obtains from (2.1) that the restriction of B^m to Λ_c preserves λ_c .

Fix a positive continuous function μ on $B^*\Gamma$ and denote by \mathcal{F} the set of all Liouville tori. For any continuous function K on Γ the mapping $\mathcal{R}_{K,\mu} : \mathcal{F} \rightarrow \mathbb{R}$, given by

$$\mathcal{R}_{K,\mu}(\Lambda_c) := \frac{1}{\lambda_c(\Lambda_c)} \int_{\Lambda_c} (\pi_\Gamma^* K) \mu \lambda_c, \quad (2.2)$$

is called a *Radon transform* of K . It is easy to see that the Radon transform does not depend on the different choices made in the definition of the Leray form.

Remark 2.1. *An alternative definition of the Radon transform would be*

$$\tilde{\mathcal{R}}_{K,\mu}(\Lambda_c) := \frac{1}{\lambda_c(\Lambda_c)} \sum_{j=0}^{m-1} \int_{\Lambda_c} (B^*)^j ((\pi_\Gamma^* K) \mu) \lambda_c \quad (2.3)$$

where $m \geq 1$ is the minimal power of B that leaves Λ_c invariant, i.e., $B^m(\Lambda_c) = \Lambda_c$. Note that (2.3) appears as a spectral invariant of (1.2) in [12]. We show in Sect. 5 that for L.B.T. of classical type $m = 1$ in the charts U_1 and U_2 . In particular, (2.2) and (2.3) coincide in this case.

There is another notion of complete integrability which is related to the “billiard flow” of the billiard table (X, g) (cf. Definition 7.2). We reformulate Definition 7.2 in terms of the cotangent bundle T^*X : A billiard table is *completely integrable* if there exist n smooth functions $H_1, \dots, H_{n-1}, H_n = H$ in a neighborhood U of S^*X in T^*X with the following properties:

- (i) the functions H_j are in involution in U with respect to the canonical Poisson bracket on T^*X , i.e. $\{H_i, H_j\} = 0$, $1 \leq i, j \leq n$,
- (ii) H_1, \dots, H_n are functionally independent in U ,
- (iii) $r^*H_j = H_j$ in $U|_\Gamma$ for $1 \leq j \leq n$.

The properties (i) and (iii) imply that H_j is invariant with respect to the billiard flow in U for any $1 \leq j \leq n$. In particular, the functions $F_j = H_j \circ \pi^+$, $1 \leq j \leq n-1$, are integrals of the billiard ball map B . As H_1, \dots, H_n are functionally independent in U the billiard ball map is completely integrable if, for example, the integrals H_j are homogeneous functions with respect to the standard action of $\mathbb{R}^* := \mathbb{R} \setminus 0$ on the fibers of $T^*X \setminus 0$. In this way we see that the billiard ball map of a completely integrable billiard table is completely integrable if the integrals are homogeneous functions on the fibers of $T^*X \setminus 0$.

Definition 2.2. *A billiard table (X, g) with a completely integrable billiard ball map will be called \mathcal{R} -rigid with respect to the density μ if Problem B has a positive solution.*

3 Liouville billiard tables

3.1 Construction of Liouville billiard tables

In this section we describe a class of 3-dimensional completely integrable billiard tables called *Liouville billiard tables*. The interior of an ellipsoid is a particular case of a Liouville billiard table – see § 3.2 below as well as § 5.3 in [11] for the general construction of Liouville billiard tables of arbitrary dimension, where the integrability of the billiard ball map was deduced from geodesically equivalence principle. Here we write explicitly integrals of the billiard flow of a Liouville billiard table which are quadratic forms in momenta, and hence, homogeneous functions of degree 2 on the fibers of $T^*X \setminus 0$.

For any $N > 0$ and any $\omega_k > 0$ ($k = 1, 2$) consider the cylinder

$$C := \{(\theta_1, \theta_2, \theta_3)\} \cong \mathbb{T}_{\omega_1} \times \mathbb{T}_{\omega_2} \times [-N, N], \quad \mathbb{T}_l := \mathbb{R}/l\mathbb{Z},$$

where θ_1 and θ_2 are periodic coordinates with minimal periods ω_1 and ω_2 respectively and θ_3 takes its values in the closed interval $[-N, N]$. Define the involutions $\sigma_1, \sigma_2 : C \rightarrow C$ of the cylinder C by

$$\sigma_1 : (\theta_1, \theta_2, \theta_3) \mapsto (-\theta_1, \frac{\omega_2}{2} - \theta_2, \theta_3) \tag{3.1}$$

and

$$\sigma_2 : (\theta_1, \theta_2, \theta_3) \mapsto (\theta_1, -\theta_2, -\theta_3). \tag{3.2}$$

As the commutator $[\sigma_1, \sigma_2]$ vanishes one can define the action of the Abelian group $\mathcal{A} := \mathbb{Z}_2 \oplus \mathbb{Z}_2$ on C by $(\alpha, \theta) \mapsto \alpha \cdot \theta := (\sigma_1^{\alpha_1} \circ \sigma_2^{\alpha_2})(\theta)$, where $\alpha = (\alpha_1, \alpha_2) \in \mathcal{A}$ and $\theta \in C$. Consider the equivalence relation $\sim_{\mathcal{A}}$ on C defined as follows: The points $p, q \in C$ are equivalent $p \sim_{\mathcal{A}} q$ iff they belong to the same orbit of \mathcal{A} (i.e., there is $\alpha \in \mathcal{A}$ such that $\alpha \cdot p = q$). Denote by \tilde{C} the topological quotient $C / \sim_{\mathcal{A}}$ of C with respect to the action of \mathcal{A} and let

$$\sigma : C \rightarrow \tilde{C} \tag{3.3}$$

be the corresponding projection. A point $p \in C$ is called a *regular* point of the projection (3.3) iff it is not a fixed point of the action for any $0 \neq \alpha \in \mathcal{A}$. The points in C that are not regular will be called *singular* or *branched points* of the projection σ . The set of singular points is given by $S := S_1 \sqcup S_2$, where

$$S_1 := \left\{ \left(\theta_1 \equiv 0 \left(\bmod \frac{\omega_1}{2} \right), \theta_2 \equiv \frac{\omega_2}{4} \left(\bmod \frac{\omega_2}{2} \right), \theta_3 \right) : -N \leq \theta_3 \leq N \right\}$$

and

$$S_2 := \left\{ \left(\theta_1, \theta_2 \equiv 0 \left(\bmod \frac{\omega_2}{2} \right), \theta_3 = 0 \right) : \theta_1 \in \mathbb{T}_{\omega_1} \right\}.$$

The set $S_1 \subset C$ has four connected components homeomorphic to the unit interval $[0, 1] \subset \mathbb{R}$ while $S_2 \subset C$ has two connected components homeomorphic to \mathbb{T} .

Lemma 3.1. *The space \tilde{C} is homeomorphic to the unit disk \mathbb{D}^3 in \mathbb{R}^3 . The map $\sigma : C \rightarrow \tilde{C}$ is a 4-folded branched covering of \tilde{C} .*

Remark 3.2. *The image \tilde{S}_1 of S_1 under the projection $\sigma : C \rightarrow \tilde{C}$ is homeomorphic to the disjoint union of two unit intervals and the image \tilde{S}_2 of S_2 is homeomorphic to \mathbb{T} .*

Proof of Lemma 3.1. First consider the action of the involution σ_1 on the cylinder

$$C = \mathbb{T}_{\omega_1} \times \mathbb{T}_{\omega_2} \times [-N, N].$$

For any value $c \in [-N, N]$ the involution σ_1 is acting on the 2-torus $\mathbb{T}_c^2 := \mathbb{T}_{\omega_1} \times \mathbb{T}_{\omega_2} \times \{\theta_3 = c\}$ by

$$\sigma_1(c) : (\theta_1, \theta_2) \mapsto \left(-\theta_1, \frac{\omega_2}{2} - \theta_2 \right).$$

The involution $\sigma_1(c) : \mathbb{T}_c^2 \rightarrow \mathbb{T}_c^2$ has four fixed points and it is easy to see that the topological quotient $\mathbb{S}^2(c)$ of \mathbb{T}_c^2 with respect to the orbits of the action of $\sigma_1(c)$ is homeomorphic to the 2-sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x|^2 = 1\}$. Hence,

$$C_1 := (C / \sim_{\sigma_1}) \cong \{(\mathbb{S}^2(c), c) : c \in [-N, N]\}. \quad (3.4)$$

Under the identification (3.4), the involution $\sigma_2 : C_1 \rightarrow C_1$ becomes

$$\begin{aligned} \mathbb{S}^2 \times [-N, N] &\rightarrow \mathbb{S}^2 \times [-N, N] \\ (x_1, x_2, x_3; c) &\mapsto (x_1, x_2, -x_3; -c). \end{aligned}$$

The fixed points of this involution form a submanifold, $\{(x_1, x_2, 0; 0) : x_1^2 + x_2^2 = 1\} \cong \mathbb{T}$, and the corresponding quotient is homeomorphic to \mathbb{D}^3 , hence, $\tilde{C} \cong (C_1 / \sim_{\sigma_2}) \cong \mathbb{D}^3$. \square

In what follows we will define a differential structure \mathcal{D} on \tilde{C} and a smooth Riemannian metrics \tilde{g} on the manifold $X := (\tilde{C}, \mathcal{D}) \cong \mathbb{D}^3$ such that the billiard table (X, \tilde{g}) becomes completely integrable. The branched covering $\sigma : C \rightarrow \tilde{C}$ defined above will play an important role in our construction. To this end choose three real-valued C^∞ -smooth functions $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi_3 : [-N, N] \rightarrow \mathbb{R}$ satisfying the following properties:

(A₁) φ_k ($k = 1, 2, 3$) is an even function depending only on the variable θ_k and $\varphi_1(\theta_1) \geq \varphi_2(\theta_2) \geq 0 \geq \varphi_3(\theta_3)$;

φ_k ($k = 1, 2$) is periodic with period ω_k ;

φ_2 satisfies the additional symmetry $\varphi_2(\theta_2) = \varphi_2\left(\frac{\omega_2}{2} - \theta_2\right)$,

(A₂) $\nu_k := \min \varphi_k = \max \varphi_{k+1}$ ($k = 1, 2$) and $\nu_1 > \nu_2 = 0$;

for any $k \in \{1, 2\}$, $\varphi_k(\theta_k) = \nu_k$ iff $\theta_k \equiv 0 \pmod{\frac{\omega_k}{2}}$;

$\varphi_2(\theta_2) = \nu_1$ iff $\theta_2 \equiv \frac{\omega_2}{4} \pmod{\frac{\omega_2}{2}}$;

$\varphi_3(\theta_3) = \nu_2$ iff $\theta_3 = 0$;

(A₃) compatibility conditions:

(1) for $k \in \{1, 2\}$, $\varphi_k''(0) = \varphi_k''(\omega_k/2) > 0$ and $\varphi_1^{(2l)}(0) = \varphi_1^{(2l)}(\omega_1/2)$;²

(2) for any $l \geq 0$, $\varphi_1^{(2l)}(0) = (-1)^l \varphi_2^{(2l)}(\omega_2/4)$ and $\varphi_2^{(2l)}(0) = (-1)^l \varphi_3^{(2l)}(0)$.

²Item (A₅) (i) in [11] has to be written similarly.

Consider the following quadratic forms on TC (quadratic on any fiber $T_\theta C$)

$$dg^2 := \Pi_1 d\theta_1^2 + \Pi_2 d\theta_2^2 + \Pi_3 d\theta_3^2 \quad (3.5)$$

and

$$\begin{aligned} dI_1^2 &:= (\varphi_2 + \varphi_3)\Pi_1 d\theta_1^2 + (\varphi_1 + \varphi_3)\Pi_2 d\theta_2^2 + (\varphi_1 + \varphi_2)\Pi_3 d\theta_3^2, \\ dI_2^2 &:= (\varphi_2\varphi_3)\Pi_1 d\theta_1^2 + (\varphi_2\varphi_3)\Pi_2 d\theta_2^2 + (\varphi_2\varphi_3)\Pi_3 d\theta_3^2, \end{aligned} \quad (3.6)$$

where

$$\Pi_1 := (\varphi_1 - \varphi_2)(\varphi_1 - \varphi_3), \quad \Pi_2 := (\varphi_1 - \varphi_2)(\varphi_2 - \varphi_3), \quad \text{and} \quad \Pi_3 := (\varphi_1 - \varphi_3)(\varphi_2 - \varphi_3).$$

We say also that the forms above are quadratic forms on C . Notice that dg^2 is *degenerate*, it vanishes on S .

Proposition 3.3. *Assume that the functions $\varphi_1, \varphi_2, \varphi_3$ satisfy $(A_1) \div (A_3)$. Then there exists a differential structure \mathcal{D} on \tilde{C} such that the projection $\sigma : C \rightarrow \tilde{C}$ is smooth and σ is a local diffeomorphism in the regular points. The push-forwards $\tilde{g} := \sigma_* g$ and $\tilde{I}_k := \sigma_* I_k$ ($k = 1, 2$) are smooth quadratic forms and \tilde{g} is a Riemannian metric on $X := (\tilde{C}, \mathcal{D})$. In addition, the billiard table (X, \tilde{g}) is completely integrable and the quadratic forms \tilde{I}_1, \tilde{I}_2 , and $\tilde{I}_3(\xi) := \tilde{g}(\xi, \xi)/2$, considered as functions on TX are functionally independent and Poisson commuting³ integrals of the billiard flow of \tilde{g} .*

Proof of Proposition 3.3. Consider the set $S = S_1 \sqcup S_2 \subset C$ of branched points of the covering $\sigma : C \rightarrow \tilde{C}$. Take a point $p = (\theta_1^0, \theta_2^0, \theta_3^0) \in S_1$ and assume for example that $\theta_1^0 = 0, \theta_2^0 = \frac{\omega_2}{4}$ and $\theta_3^0 \in [-N, N]$. Define a new chart $V_1 = \{(x_1, x_2, x_3)\}$ in a neighborhood of p by $x_k := \theta_k - \theta_k^0$, $k = 1, 2$, and $x_3 := \theta_3$, where $|x_k| < \omega_k/8$ for $k = 1, 2$ and $|x_3| \leq N$. In this chart $p = (0, 0, \theta_3^0)$ and

$$dg^2 = Q_1 dx_1^2 + Q_2 dx_2^2 + Q_3 dx_3^2, \quad (3.7)$$

$$dI_1^2 = (\phi_2 + \phi_3)Q_1 dx_1^2 + (\phi_1 + \phi_3)Q_2 dx_2^2 + (\phi_1 + \phi_2)Q_3 dx_3^2, \quad (3.8)$$

$$dI_2^2 = (\phi_2\phi_3)Q_1 dx_1^2 + (\phi_1\phi_3)Q_2 dx_2^2 + (\phi_1\phi_2)Q_3 dx_3^2, \quad (3.9)$$

where $Q_1 := (\phi_1 - \phi_2)(\phi_1 - \phi_3)$, $Q_2 := (\phi_1 - \phi_2)(\phi_2 - \phi_3)$, $Q_3 := (\phi_1 - \phi_3)(\phi_2 - \phi_3)$, $\phi_k(x_k) = \varphi_k(\theta_k^0 + x_k)$, $k = 1, 2$, and $\phi_3(x_3) = \varphi_3(x_3)$. Note that V_1 is a tubular neighborhood of the chosen component of S_1 and it does not intersect the other components of S . It follows from $(A_1) \div (A_3)$ that the functions ϕ_1, ϕ_2 , and ϕ_3 are smooth and have the following properties in V_1 :

(L₁) ϕ_k is even and depends only on the variable x_k ;

(L₂) $\phi_1 > \phi_3$ and $\phi_2 > \phi_3$;

(L₃) ϕ_1 and ϕ_2 satisfy:

- (i) $\phi_1 > \nu_1$ if $x_1 \neq 0$ and $\phi_1(0) = \nu_1, \phi_1''(0) > 0$;
- (ii) $\phi_2 < \nu_1$ if $x_2 \neq 0$ and $\phi_2(0) = \nu_1, \phi_2''(0) < 0$;

³The canonical symplectic structure on T^*X induces a symplectic structure on TX by identifying vectors and covectors by means of the Riemannian metric \tilde{g} .

(iii) $\phi_1^{(2l)}(0) = (-1)^l \phi_2^{(2l)}(0)$ for any $l \geq 0$.

In the new coordinates, the involution $\sigma_1|_{V_1}$ becomes $\sigma_1|_{V_1} : (x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$. In order to define a differential structure in a neighborhood of $\sigma(p)$ in \tilde{C} consider the mapping $\Phi_1 : V_1 \rightarrow \text{Im } \Phi_1 := W_1$,

$$\Phi_1 : (x_1, x_2, x_3) \mapsto (y_1 = x_1^2 - x_2^2, y_2 = 2x_1 x_2, y_3 = x_3). \quad (3.10)$$

By Lemma 3.4 below the push-forwards $\tilde{g}|_{W_1} := \Phi_{1*}(g|_{V_1})$, $\tilde{I}_1|_{W_1} := \Phi_{1*}(I_1|_{V_1})$, and $\tilde{I}_2|_{W_1} := \Phi_{1*}(I_2|_{V_1})$ are smooth quadratic forms on W_1 and $\tilde{g}|_{W_1}$ is positive definite. Since $\Phi_1 \circ (\sigma_1|_{V_1}) = \Phi_1$ and $\sigma_2(V_1) \cap V_1 = \emptyset$ we can identify $\sigma|_{V_1}$ with Φ_1 and get a differential structure in the neighborhood of $\sigma(p) \in \tilde{C}$. In a similar way we construct a tubular neighborhood V_2 of the component $\theta_1^0 = \omega_2/2$, $\theta_2^0 = \frac{\omega_2}{4}$ and $\theta_3^0 \in [-N, N]$ of S_1 together with a mapping $\Phi_2 : V_2 \rightarrow W_2$ such that the push-forward of $g|_{V_2}$, $I_1|_{V_2}$, and $I_2|_{V_2}$ are smooth quadratic forms on W_2 . Consider also the tubular neighborhoods $V_3 := \sigma_2(V_1)$ and $V_4 := \sigma_2(V_2)$ of the other two components of S_1 in \tilde{C} together with the mappings

$$\Phi_3 := \Phi_1 \circ (\sigma_2|_{V_3}) : V_3 \rightarrow W_1 \quad \text{and} \quad \Phi_4 := \Phi_2 \circ (\sigma_2|_{V_4}) : V_4 \rightarrow W_2. \quad (3.11)$$

For $j = 3, 4$ one has $\Phi_j \circ (\sigma_1|_{V_j}) = \Phi_j$, and therefore we can identify Φ_j with $\sigma|_{V_j}$. As the quadratic forms (3.5) and (3.6) are invariant with respect to σ_2 we obtain from (3.11) that $\Phi_{3*}(g|_{V_3}) = \tilde{g}|_{W_1}$, $\Phi_{4*}(g|_{V_4}) = \tilde{g}|_{W_2}$, $\Phi_{3*}(I_j|_{V_3}) = \tilde{I}_j|_{W_1}$, and $\Phi_{4*}(I_j|_{V_4}) = \tilde{I}_j|_{W_2}$, $j = 1, 2$. In particular, the mappings $\sigma|_{V_3} : V_3 \rightarrow W_1$ and $\sigma|_{V_4} : V_4 \rightarrow W_2$ and the push-forward of (3.5) and (3.6) with respect to them are smooth. Arguing similarly we treat the case $p \in S_2$ and construct a coordinate chart W_3 of $\tilde{S}_2 = \sigma(S_2)$ in \tilde{C} .

Covering the image of the branched points of σ by the charts W_1 , W_2 , and W_3 we get a differential structure on $\sqcup_{j=1}^3 W_j \supset \tilde{S}_1 \sqcup \tilde{S}_2$. As the set $\tilde{C} \setminus (\tilde{S}_1 \sqcup \tilde{S}_2)$ consists of regular points of σ we can induce a differential structure on it from the differential structure of the cylinder C . The union of these two differential structures is compatible and defines a differential structure \mathcal{D} on \tilde{C} . Denote by X the smooth manifold $X = (\tilde{C}, \mathcal{D})$. It follows from (A₁) that the forms (3.5) and (3.6) on C are invariant under the involutions (3.1) and (3.2). In particular, the push-forwards $\tilde{g} := \sigma_* g$, $\tilde{I}_1 := \sigma_* I_1$, and $\tilde{I}_2 := \sigma_* I_2$ are smooth quadratic forms on $X \setminus (\tilde{S}_1 \sqcup \tilde{S}_2)$. Moreover, we have seen that the push-forwards \tilde{g} , \tilde{I}_1 , and \tilde{I}_2 are smooth quadratic forms on W_j , and that \tilde{g} is a Riemannian metric in W_j for any $j \in \{1, 2, 3\}$. Hence, the push-forwards \tilde{g} , \tilde{I}_1 , and \tilde{I}_2 are smooth quadratic forms on X and \tilde{g} is a Riemannian metric. We will show that I_1 and I_2 are integrals of the billiard flow of the metric g on $C \setminus S$. Indeed, applying the Legendre transformation $p_k = \Pi_k \dot{\theta}_k$, $k = 1, 2, 3$ (which is well defined only on $C \setminus S$) and dropping for simplicity the factor $\frac{1}{2}$ in the Hamiltonian we get

$$\left\{ \begin{array}{lcl} H & = & \frac{1}{\Pi_1} p_1^2 + \frac{1}{\Pi_2} p_2^2 + \frac{1}{\Pi_3} p_3^2 \\ I_1 & = & \frac{\varphi_2 + \varphi_3}{\Pi_1} p_1^2 + \frac{\varphi_1 + \varphi_3}{\Pi_2} p_2^2 + \frac{\varphi_1 + \varphi_2}{\Pi_3} p_3^2 \\ I_2 & = & \frac{(\varphi_2 \varphi_3)}{\Pi_1} p_1^2 + \frac{(\varphi_1 \varphi_3)}{\Pi_2} p_2^2 + \frac{(\varphi_1 \varphi_2)}{\Pi_3} p_3^2 \end{array} \right. \quad (3.12)$$

which can be rewritten in *Stäkel form* (cf. [14], [13, § 2])

$$\begin{cases} p_1^2 = \varphi_1^2 H - \varphi_1 I_1 + I_2 \\ p_2^2 = -\varphi_2^2 H + \varphi_2 I_1 - I_2 \\ p_3^2 = \varphi_3^2 H - \varphi_3 I_1 + I_2 \end{cases} . \quad (3.13)$$

In particular, the functions H , I_1 , and I_2 Poisson commute with respect to the canonical symplectic form $\omega := dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2 + dp_3 \wedge d\theta_3$ on the cotangent bundle $T^*(C \setminus S)$ (see for example [13, Proposition 1]). Moreover, the forms I_1 and I_2 are invariant with respect to the reflection map at the boundary $\rho : (TC)|_{\partial C} \rightarrow (TC)|_{\partial C}$ given by

$$(\theta_1, \theta_2, \pm N, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) \xrightarrow{\rho} (\theta_1, \theta_2, \pm N, \dot{\theta}_1, \dot{\theta}_2, -\dot{\theta}_3).$$

Hence \tilde{I}_1 and \tilde{I}_2 are Poisson commuting integrals of the billiard flow of the metric \tilde{g} on $X \setminus \sigma(S)$. As $\sigma(S)$ is a 1-dimensional submanifold in the 3-manifold X we get that \tilde{I}_1 and \tilde{I}_2 are Poisson commuting integrals of the billiard flow of the metric \tilde{g} . A direct computation shows that H , I_1 and I_2 in (3.12) are functionally independent on $T^*(C \setminus S)$. Hence, \tilde{H} , \tilde{I}_1 and \tilde{I}_2 are functionally independent on $T^*(X \setminus \sigma(S))$. \square

Lemma 3.4 . *The quadratic forms $\tilde{g}|_{W_1} = \Phi_{1*}(g|_{V_1})$, $\tilde{I}_1|_{W_1} = \Phi_{1*}(I_1|_{V_1})$, and $\tilde{I}_2|_{W_1} = \Phi_{1*}(I_2|_{V_1})$ are smooth and $\tilde{g}|_{W_1}$ is positive definite.*

Proof of Lemma 3.4. A direct computation involving (3.10) shows that

$$d\tilde{g}^2|_{(W_1 \setminus \tilde{S}_1)} = \tilde{g}_{11} dy_1^2 + 2\tilde{g}_{12} dy_1 dy_2 + \tilde{g}_{22} dy_2^2 + \tilde{g}_{33} dy_3^2$$

where

$$\tilde{g}_{11} = \frac{1}{4} \left(\frac{\phi_1 - \phi_2}{x_1^2 + x_2^2} \right) \left(\frac{(\phi_1 - \phi_3)x_1^2 + (\phi_2 - \phi_3)x_2^2}{x_1^2 + x_2^2} \right), \quad \tilde{g}_{12} = \frac{1}{4} \left(\frac{\phi_1 - \phi_2}{x_1^2 + x_2^2} \right)^2 x_1 x_2, \quad (3.14)$$

and

$$\tilde{g}_{22} = \frac{1}{4} \left(\frac{\phi_1 - \phi_2}{x_1^2 + x_2^2} \right) \left(\frac{(\phi_1 - \phi_3)x_2^2 + (\phi_2 - \phi_3)x_1^2}{x_1^2 + x_2^2} \right), \quad \tilde{g}_{33} = (\phi_1 - \phi_3)(\phi_2 - \phi_3). \quad (3.15)$$

Let $A := \phi_1 \frac{\partial}{\partial x_1} \otimes dx_1 + \phi_2 \frac{\partial}{\partial x_2} \otimes dx_2 + \phi_3 \frac{\partial}{\partial x_3} \otimes dx_3 \in C^\infty(TV_1 \otimes T^*V_1)$. A similar computation as above shows that

$$\tilde{A}|_{(W_1 \setminus \tilde{S}_1)} = \tilde{A}_{11} \frac{\partial}{\partial y_1} \otimes dy_1 + \tilde{A}_{12} \frac{\partial}{\partial y_1} \otimes dy_2 + \tilde{A}_{21} \frac{\partial}{\partial y_2} \otimes dy_1 + \phi_3 \frac{\partial}{\partial y_3} \otimes dy_3$$

where

$$\tilde{A}_{11} = \frac{\phi_1 x_1^2 + \phi_2 x_2^2}{x_1^2 + x_2^2}, \quad \tilde{A}_{12} = \frac{\phi_1 - \phi_2}{x_1^2 + x_2^2} x_1 x_2, \quad \tilde{A}_{21} = \frac{\phi_1 - \phi_2}{x_1^2 + x_2^2} x_1 x_2, \quad \tilde{A}_{22} = \frac{\phi_1 x_2^2 + \phi_2 x_1^2}{x_1^2 + x_2^2}. \quad (3.16)$$

Consider the tensor field A as a section in $\text{Hom}(T^*V_1, T^*V_1)$. Then we have

$$\det(A + c)g((A + c)^{-1}\xi, \xi) = c^2g(\xi, \xi) + cI_1(\xi, \xi) + I_2(\xi, \xi) \quad (3.17)$$

for any $c > -\max_{0 \leq \theta_1 \leq \omega_1} \varphi_1(\theta_1)$. We will show that the coefficients (3.14), (3.15), and (3.16), when re-expressed in terms of the variables (y_1, y_2, y_3) , are smooth in W_1 . Then the statement of the Lemma will follow from the relation (3.17) and the properties of the Vandermonde determinant.

Consider, for example, the function

$$\Phi(x_1, x_2) := \frac{\phi_1(x_1) - \phi_2(x_2)}{x_1^2 + x_2^2}, \quad (x_1, x_2) \neq (0, 0).$$

Fix $m \in \mathbb{N}$, $m \geq 1$. Using (L_3) and the Taylor formula with an integral reminder term we get

$$\begin{aligned} \phi_1(x_1) &= \sum_{k=0}^m a_k x_1^{2k} + x_1^{2m+1} S_{1,2m+1}(x_1) \\ \phi_2(x_2) &= \sum_{k=0}^m (-1)^k a_k x_2^{2k} + x_2^{2m+1} S_{2,2m+1}(x_2) \end{aligned}$$

where $S_{j,2m+1}$, $j = 1, 2$, are smooth functions in a neighborhood of 0. Lemma 3.5 below implies that

$$\Phi(x_1, x_2) = \sum_{k=0}^{m-1} \Phi_k(y_1, y_2) + S_{2m+1}(x_1, x_2) \quad \text{for } (x_1, x_2) \neq (0, 0),$$

where $\Phi_k(y_1, y_2) := P_k(y_1, y_2)$ for k -odd and $\Phi_k(y_1, y_2) := R_k(y_1, y_2)$ for k -even are homogeneous polynomials of degree $2k$ with respect to (y_1, y_2) , and

$$S_{2m+1}(x_1, x_2) := \frac{x_1^{2m+1} S_{1,2m+1}(x_1) - x_2^{2m+1} S_{2,2m+1}(x_2)}{x_1^2 + x_2^2}, \quad (x_1, x_2) \neq (0, 0).$$

Consider the directional derivatives

$$\partial_{y_1} := \frac{\partial}{\partial y_1} = \frac{1}{2(x_1^2 + x_2^2)} \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right) \text{ and } \partial_{y_2} := \frac{\partial}{\partial y_2} = \frac{1}{2(x_1^2 + x_2^2)} \left(x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right).$$

We have

$$\lim_{(x_1, x_2) \rightarrow (0, 0)} \partial_{y_1}^\alpha \partial_{y_2}^\beta S_{2m+1}(x_1, x_2) = 0$$

for $\alpha + \beta \leq m$. Hence, Φ can be extended by continuity to a C^∞ -smooth function in the variables (y_1, y_2) in a neighborhood of $(0, 0)$ and its Taylor series is $\sum_{k=0}^\infty \Phi_k(y_1, y_2)$. In the case when ϕ_1 and ϕ_2 are real analytic the power series $\sum_{k=0}^\infty \Phi_k(y_1, y_2)$ is uniformly convergent in a neighborhood of $(0, 0)$.

Arguing similarly we obtain that the coefficients (3.14)-(3.16) are C^∞ -smooth in the variables (y_1, y_2) when ϕ_1 and ϕ_2 are smooth and real analytic if ϕ_1 and ϕ_2 are real analytic. Moreover, by Taylor's formula $\phi_1(x_1) = \nu_1 + a_1 x_1^2 + o(x_1^2)$ as $x_1 \rightarrow 0$ and $\phi_2(x_2) = \nu_1 - a_1 x_2^2 + o(x_2^2)$ as $x_2 \rightarrow 0$ that together with (3.14) and (3.15) implies $\tilde{g}_{11} = a_1(\nu_1 - \phi_3) + o(1)$, $\tilde{g}_{12} = o(1)$, and $\tilde{g}_{22} = a_1(\nu_1 - \phi_3) + o(1)$ as $y \rightarrow ((0, 0, y_3^0))$. Hence, $d\tilde{g}^2|_{(W_1 \setminus \tilde{S}_1)}$ can be extended by continuity to $(0, 0, y_3^0) \in \tilde{S}_1$ and by (L_2) the extension is positive definite. This completes the proof of the Lemma. \square

Lemma 3.5. *For any $m \geq 2$,*

$$x_1^{2m} - x_2^{2m} = \begin{cases} (x_1^2 + x_2^2) P_{m-1}(y_1, y_2), & m - \text{even}, \\ Q_m(y_1, y_2), & m - \text{odd}, \end{cases}$$

$$x_1^{2m} + x_2^{2m} = \begin{cases} (x_1^2 + x_2^2) R_{m-1}(y_1, y_2), & m - \text{odd}, \\ N_m(y_1, y_2), & m - \text{even}, \end{cases}$$

where P_m , Q_m , R_m , and N_m are polynomials of y_1 and y_2 of degree m .

Proof of Lemma 3.5. Introduce the complex variables $z := x_1 + ix_2$ and $w := y_1 + iy_2$ and note that $w = z^2$. Then, for any $m \geq 2$, $x_1^{2m} \pm x_2^{2m} = ((z + \bar{z})^{2m} \pm (-1)^m(z - \bar{z})^{2m})/2^{2m}$. Finally, using Newton's binomial formula one concludes the Lemma. \square

Following [10] we impose the following additional assumptions on the functions φ_k :

$$(A_4) \quad \varphi_1(\theta_1) = \varphi_1(\omega_1/2 - \theta_1)$$

$$(A_5) \quad \text{for any } k \in \{1, 2\} \text{ the derivative } \varphi'_k(\theta_k) > 0 \text{ on } (0, \omega_k/4) \text{ and } \varphi'_3(\theta_3) < 0 \text{ on } (0, N].$$

The condition $\varphi'_3(N) < 0$ means that the boundary of X is locally geodesically convex.

Definition 3.6. *The billiard table (X, \tilde{g}) in Proposition 3.3 is called a Liouville billiard table (shortly L.B.T.). Liouville billiard tables satisfying conditions (A_4) and (A_5) are called Liouville billiard tables of classical type. In the case when φ_1 , φ_2 , and φ_3 are real analytic, the billiard table is called analytic L.B.T.*

The involutions,

$$\begin{aligned} (\theta_1, \theta_2, \theta_3) &\mapsto (-\theta_1, \theta_2, \theta_3) \\ (\theta_1, \theta_2, \theta_3) &\mapsto \left(\frac{\omega_1}{2} - \theta_1, \theta_2, \theta_3\right) \\ (\theta_1, \theta_2, \theta_3) &\mapsto (\theta_1, -\theta_2, \theta_3) \end{aligned} \tag{3.18}$$

induce a group of isometries $G(X) = G(X, \tilde{g})$ on X which is isomorphic to the direct sum

$$G(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Remark 3.7. *The action of $G(X)$ on (X, \tilde{g}) is an analog of the action of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the interior of the ellipsoid in $\mathbb{R}^3 = \{(x, y, z)\}$ generated by the reflections with respect to the coordinate planes O_{xy} , O_{yz} and O_{xz} .*

Remark 3.8. *The compatibility conditions $\varphi_1^{(2l)}(0) = \varphi_1^{(2l)}(\omega_1/2)$, $l = 0, 1, \dots$, in (A_3) follows from (A_4) for L.B.T.s of classical type.*

3.2 Ellipsoidal billiard tables

Denote by \mathbb{R}^3 the Euclidean space $\mathbb{R}^3 = \{(x_1, x_2, x_3)\}$ supplied with the standard Euclidean metric $dg_0^2 := dx_1^2 + dx_2^2 + dx_3^2$. A class of L.B.T.s in \mathbb{R}^3 depending on 3 real parameters $b_1 > b_2 > b_3$ can be obtained using the mapping:

$$\Sigma_0 : \begin{cases} x_1 = \sqrt{(b_1 - \lambda_2) \frac{b_1 - \lambda_3}{b_1 - b_3}} \cos \phi_1 \\ x_2 = \sqrt{b_2 - b_3} \sin \phi_1 \cos \phi_2 \\ x_3 = \sqrt{\frac{b_3 - \lambda_1}{b_3 - b_1}} \phi_3 \sin \phi_2 \end{cases}$$

where $\lambda_k := b_{k+1} + (b_k - b_{k+1}) \sin^2 \phi_k$ ($k = 1, 2$), $\lambda_3 := b_3 - \phi_3^2$, ϕ_k ($k = 1, 2$) are periodic coordinates with period 2π , and $-N \leq \phi_3 \leq N$. The mapping $\Sigma_0 : \mathbb{T}^2 \times [-N, N] \rightarrow \mathbb{R}^3$ gives a 4-folded branched covering of an ellipsoidal domain X in \mathbb{R}^3 and (X, g_0) is a L.B.T. of classical type – for details see § 5 in [11]. More generally, the two-parameter family of billiard tables $(M^3, g_{\alpha, \beta})$ of constant scalar curvature κ in [11, Theorem 3] consists of L.B.T.s of classical type according to § 5.4 in [11]. The boundary of any billiard table of the family is geodesically equivalent to the ellipsoid. In particular, it has non-periodic geodesics and satisfies the hypothesis of Theorem 1 and Theorem 3. This family contains the ellipsoid ($\kappa = 0$) and L.B.T.s of both positive and negative scalar curvature that are realized on the standard sphere and on the hyperbolic space respectively.

3.3 Parameterization of the Lagrangian tori

The aim of this section is to obtain charts of action-angle coordinates for L.B.T.s of classical type and to parameterize the corresponding Liouville tori. Recall that a L.B.T. (X, \tilde{g}) is obtained as a quotient space of the cylinder

$$C = \{(\theta_1 \pmod{\omega_1}, \theta_2 \pmod{\omega_2}, \theta_3)\} \cong \mathbb{T}_{\omega_1} \times \mathbb{T}_{\omega_2} \times [-N, N]$$

with respect to the group action of $\mathcal{A} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as described in Sect. 3.1. By Proposition 3.3, the projection $\sigma : C \rightarrow X$ is smooth and invariant with respect to the group action of \mathcal{A} on C . Moreover, the push-forwards of the quadratic forms (3.6) with respect to the projection $\sigma : C \rightarrow X$ are integrals of the billiard flow on (X, \tilde{g}) . The boundary ∂C of C has two connected components defined by $\theta_3 = \pm N$ and we set

$$\mathbb{T}_N^2 := \{(\theta_1 \pmod{\omega_1}, \theta_2 \pmod{\omega_2}, \theta_3 = N)\}.$$

By construction the restriction $\sigma|_{\mathbb{T}_N^2}$ of the projection $\sigma : C \rightarrow X$ to \mathbb{T}_N^2 is a double branched covering of the boundary $\Gamma = \partial X$.

Denote $C_r := C \setminus S$ and introduce on T^*C the coordinates $\{(\theta_1, \theta_2, \theta_3; p_1, p_2, p_3)\}$, where p_1, p_2 , and p_3 are the conjugated impulses. The Legendre transformation corresponding to the Lagrangian $L_g(\xi) := g(\xi, \xi)/2$, $\xi \in TC_r$, transforms the Lagrangian and the integrals (3.6) to the functions H , I_1 and I_2 on T^*C_r given by (3.12).⁴ Set

$$Q_1 := T^*C_r|_{\mathbb{T}_N^2} = \{\eta = (\theta, p) \in T^*C_r : \theta_3 = N\}.$$

⁴For simplicity we drop the factor $\frac{1}{2}$ in the Hamiltonian function.

The restriction $\tilde{\omega}_1$ of the symplectic two-form

$$\omega = dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2 + dp_3 \wedge d\theta_3$$

to Q_1 is $\tilde{\omega}_1 := \omega|_{Q_1} = dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2$. This form is degenerate and its kernel $\text{Ker } \tilde{\omega}_1$ is spanned on the vector field $\frac{\partial}{\partial p_3}$. Denote by Q the *isoenergy surface*

$$Q := \{\eta \in T^*C_r : H(\eta) = 1\}$$

and consider the set $Q_2 := Q \cap Q_1$. It is clear that Q_2 is diffeomorphic to the restriction of the unit cosphere bundle $S_g^*C_r$ of C_r to the torus \mathbb{T}_N^2 . The set

$$Q_2^+ := \{\eta = (\theta, p) \in Q_2 : p_3 < 0\}$$

can be identified with the set $S_+^*C_r|_{\mathbb{T}_N^2}$ of all η in $S_g^*C_r|_{\mathbb{T}_N^2}$ such that $\langle \eta, n_g \rangle > 0$, where n_g denotes the inward unit normal to $\mathbb{T}_N^2 \setminus S_1$. Moreover, the open coball bundle $B_g^*(\mathbb{T}_N^2 \setminus S_1)$ can be identified with

$$\left\{ (\theta_1, \theta_2; p_1, p_2) \in T^*(\mathbb{T}^2 \setminus S') : \left(\frac{p_1^2}{\Pi_1} + \frac{p_2^2}{\Pi_2} \right) |_{\theta_3=N} < 1 \right\}, \quad (3.19)$$

where $S' := \{(\theta_1 \equiv 0 \pmod{\frac{\omega_1}{2}}, \theta_2 \equiv \frac{\omega_2}{4} \pmod{\frac{\omega_2}{2}})\}$. Consider the map $R : B_g^*(\mathbb{T}_N^2 \setminus S_1) \rightarrow Q_2^+$ given by

$$R : (\theta_1, \theta_2; p_1, p_2) \mapsto (\theta_1, \theta_2; p_1, p_2, p_3) \text{ where } p_3 = -\sqrt{\Pi_3} \sqrt{1 - \frac{p_1^2}{\Pi_1} - \frac{p_2^2}{\Pi_2}}.$$

The coball bundle $B_g^*(\mathbb{T}_N^2 \setminus S_1)$ can be considered as a phase space of the billiard ball map $B : B^*(\Gamma \setminus \sigma(S)) \rightarrow B^*(\Gamma \setminus \sigma(S))$ via the branched double covering $\sigma|_{\mathbb{T}_N^2} : \mathbb{T}_N^2 \rightarrow \Gamma$. In this setting the map R can be identified with π^+ . We have also $\tilde{\omega}_2 := R^*\tilde{\omega}_1 = dp_1 \wedge d\theta_1 + dp_2 \wedge d\theta_2$. Moreover, the functions $\mathcal{I}_1 := R^*I_1$ and $\mathcal{I}_2 := R^*I_2$ are functionally independent integrals of B in $B_g^*(\mathbb{T}_N^2 \setminus S_1)$.

In the coordinates $\{(\theta_1, \theta_2; p_1, p_2)\}$ the integrals \mathcal{I}_1 and \mathcal{I}_2 become (cf. (3.12))

$$\mathcal{I}_1 = (\varphi_1 + \varphi_2) - (\varphi_1 - \nu_3) \frac{p_1^2}{\Pi_1} - (\varphi_2 - \nu_3) \frac{p_2^2}{\Pi_2}, \quad (3.20)$$

$$\mathcal{I}_2 = \varphi_1 \varphi_2 - \varphi_2 (\varphi_1 - \nu_3) \frac{p_1^2}{\Pi_1} - \varphi_1 (\varphi_2 - \nu_3) \frac{p_2^2}{\Pi_2}, \quad (3.21)$$

where $\nu_3 := \varphi_3(N) < \nu_2 = 0$ in view of (A₁) and (A₂). In order to describe the invariant manifolds of the billiard ball map B we choose real constants h_1 and h_2 and consider the level set

$$\tilde{L}_h := \{\mathcal{I}_1 = h_1, \mathcal{I}_2 = h_2\} \subset B_g^*(\mathbb{T}_N^2 \setminus S_1), \quad h = (h_1, h_2).$$

Consider the quadratic polynomial,

$$P(t) := t^2 - h_1 t + h_2 = (t - \kappa_1)(t - \kappa_2), \quad (3.22)$$

where κ_1 and κ_2 are the roots of P and $h_1 = \kappa_1 + \kappa_2$, $h_2 = \kappa_1\kappa_2$. If $(\theta_1, \theta_2; p_1, p_2) \in \tilde{L}_h$, it follows from (3.13) that

$$P(\varphi_1(\theta_1)) = \varphi_1^2(\theta_1) - h_1\varphi_1(\theta_1) + h_2 = p_1^2 \geq 0, \quad (3.23)$$

$$-P(\varphi_2(\theta_2)) = -\varphi_2^2(\theta_2) + h_1\varphi_2(\theta_2) - h_2 = p_2^2 \geq 0, \quad (3.24)$$

and

$$P(\varphi_3(N)) = P(\nu_3) = \nu_3^2 - h_1\nu_3 + h_2 \geq 0. \quad (3.25)$$

Then the set \tilde{L}_h is non-empty if and only if there is a point $(\theta_1, \theta_2) \in (\mathbb{R}/\omega_1\mathbb{Z}) \times (\mathbb{R}/\omega_2\mathbb{Z})$ such that the inequalities (3.23), (3.24), and (3.25) are satisfied. In particular, it follows from (3.23) and (3.24) that the roots $\kappa_1 \leq \kappa_2$ are real, hence, $\mathcal{D} := h_1^2 - 4h_2 \geq 0$. Moreover, $(A_1) \div (A_2)$ imply

$$\begin{cases} \nu_1 \leq \varphi_1(\theta_1) \leq \nu_0, \\ 0 = \nu_2 \leq \varphi_2(\theta_2) \leq \nu_1, \\ \nu_3 = \varphi_3(N) < \nu_2 = 0. \end{cases} \quad (3.26)$$

Then the following four cases can occur:

- (A) $\nu_3 \leq \kappa_1 \leq \nu_2 = 0$ and $0 = \nu_2 \leq \kappa_2 \leq \nu_1$;
- (B) $\nu_3 \leq \kappa_1 \leq \nu_2 = 0$ and $\nu_1 \leq \kappa_2 \leq \nu_0$;
- (C) $0 = \nu_2 \leq \kappa_1 \leq \kappa_2 \leq \nu_1$;
- (D) $0 = \nu_2 \leq \kappa_1 \leq \nu_1$ and $\nu_1 \leq \kappa_2 \leq \nu_0$.

Consider the union \tilde{U}_1 of all \tilde{L}_h in $B^*(\mathbb{T}_N^2 \setminus S_1)$ such that (A) *with strict inequalities* holds for the corresponding (κ_1, κ_2) . We will see below that any \tilde{L}_h in \tilde{U}_1 is a disjoint union of Liouville tori. In the same way we define \tilde{U}_2 corresponding to (B), \tilde{U}_3 corresponding to (C) and \tilde{U}_4 corresponding to (D). Denote $U_j := \sigma_*(\tilde{U}_j) \subset B^*\Gamma$, $j = 1, 2, 3, 4$, where σ_* is the push-forward of covectors corresponding to $\sigma : C \rightarrow X$.

Definition 3.9. *We refer to cases (A) and (B) as to boundary cases and denote $\mathcal{F}_b := U_1 \cup U_2$.*

Remark 3.10. *We will see in Sect. 5 that the billiard trajectories in T^*X issuing from $U_1 \cup U_2$ “approximate” the geodesics on the boundary Γ .*

We are going to parameterize the invariant tori belonging to the level set \tilde{L}_h . To that end we need the inverse functions of $\varphi_1|_{[0, \omega_1/4]}$ and $\varphi_2|_{[0, \omega_2/4]}$. According to $(A_1) \div (A_5)$ the function φ_1 has the following properties. It is a periodic function of period $\omega_1/2$, $\varphi_1(\omega_1/4 + \theta_1) = \varphi_1(\omega_1/4 - \theta_1)$ for any θ_1 , the map $\varphi_1 : [0, \omega_1/4] \rightarrow [\nu_1, \nu_0]$ is a homeomorphism, $\varphi'_1(\theta_1) > 0$ in the interval $(0, \omega_1/4)$, and the critical points of φ_1 at $\theta_1 = 0$ and $\theta_1 = \omega_1/4$ are non degenerate. Denote by $f_1 : [\nu_1, \nu_0] \rightarrow [0, \omega_1/4]$ the inverse map of $\varphi_1|_{[0, \omega_1/4]}$. Then f_1 is smooth in (ν_1, ν_0) , $f'_1 > 0$ in that interval, and

$$\begin{cases} f_1(x_1) = F_1^+(\sqrt{x_1 - \nu_1}) & \text{as } x_1 \rightarrow \nu_1 + 0, \\ f_1(x_1) = F_1^-(\sqrt{\nu_0 - x_1}) & \text{as } x_1 \rightarrow \nu_0 - 0, \end{cases} \quad (3.27)$$

where F_1^\mp are smooth functions in a neighborhood of 0, and

$$F_1^+(0) = 0, \quad F_1^-(0) = \omega_1/4, \quad (F_1^+)'(0) = \sqrt{2\varphi_1''(0)^{-1}} \text{ and } (F_1^-)'(0) = -\sqrt{-2\varphi_1''(\omega_1/4)^{-1}}. \quad (3.28)$$

The function $\varphi_2|_{[0,\omega_2/4]}$ has the same properties, and we denote by $f_2 : [0, \nu_1] \longrightarrow [0, \omega_2/4]$ its inverse function. Then f_2 is smooth in $(0, \nu_1)$ and $f'_2 > 0$ in that interval, and

$$\begin{cases} f_2(x_2) = F_2^+(\sqrt{x_2}) & \text{as } x_2 \rightarrow 0+0, \\ f_2(x_2) = F_2^-(\sqrt{\nu_1 - x_2}) & \text{as } x_2 \rightarrow \nu_1-0, \end{cases} \quad (3.29)$$

where F_2^\mp are smooth functions in a neighborhood of 0 and

$$F_2^+(0) = 0, \quad F_2^-(0) = \omega_2/4, \quad (F_2^+)'(0) = \sqrt{2\varphi_2''(0)^{-1}}, \quad (F_2^-)'(0) = -\sqrt{-2\varphi_2''(\omega_2/4)^{-1}}. \quad (3.30)$$

Assume that $\tilde{L}_h \subset \tilde{U}_1$. We have $\nu_3 < \kappa_1 < 0$ and $0 < \kappa_2 < \nu_1$. It follows from (3.23)-(3.24) and (3.26) that \tilde{L}_h consists of four connected components $T_h^{(k)}$ ($1 \leq k \leq 4$) which are diffeomorphic to \mathbb{T}^2 . Moreover, the image of each $T_h^{(k)}$ with respect to the bundle projection $T^*\mathbb{T}_N^2 \rightarrow \mathbb{T}_N^2$ coincides with one of the annuli

$$A'_h := \{0 \leq \theta_1 \leq \omega_1; -f_2(\kappa_2) \leq \theta_2 \leq f_2(\kappa_2)\}$$

and

$$A''_h := \{0 \leq \theta_1 \leq \omega_1; \omega_2/2 - f_2(\kappa_2) \leq \theta_2 \leq \omega_2/2 + f_2(\kappa_2)\}.$$

Assume that the tori $T_h^{(1)}$ and $T_h^{(2)}$ are projected onto A'_h and similarly, $T_h^{(3)}$ and $T_h^{(4)}$ are projected onto A''_h . As the map $\sigma|_{\mathbb{T}_N^2} : \mathbb{T}_N^2 \rightarrow \Gamma$ is invariant with respect to the involution

$$\iota : (\theta_1, \theta_2) \mapsto (-\theta_1, \omega_2/2 - \theta_2),$$

and $\iota(A'_h) = A''_h$, the pairs $(T_h^{(1)}, T_h^{(2)})$ and $(T_h^{(3)}, T_h^{(4)})$ correspond the same pair of invariant tori in $T^*\Gamma$, which we identify with $(T_h^{(1)}, T_h^{(2)})$. It follows from (3.23), (3.24) and (3.26) that the map $r_{\epsilon_1 \epsilon_2} : A'_h \rightarrow T_h^{(1)}$ defined by,

$$(\theta_1, \theta_2) \xrightarrow{r_{\epsilon_1 \epsilon_2}} \left(\theta_1, \theta_2; \epsilon_1 \sqrt{\varphi_1(\theta_1)^2 - h_1 \varphi_1(\theta_1) + h_2}, \epsilon_2 \sqrt{-\varphi_2(\theta_2)^2 + h_1 \varphi_2(\theta_2) - h_2} \right), \quad (3.31)$$

gives a parametrization of the torus $T_h^{(1)}$ for $\epsilon_1 = 1$ and $\epsilon_2 = \pm 1$. In the same way, taking $\epsilon_1 = -1$ and $\epsilon_2 = \pm 1$ we parametrize $T_h^{(2)}$.

In the same way one treats the cases (B), (C) and (D). In particular, one gets that \tilde{U}_1, \tilde{U}_2 , and \tilde{U}_3 have 4 connected components while \tilde{U}_4 has 8 connected components. Similarly, U_1, U_2 , and U_3 have 2 connected components and U_4 has 4 connected components.

4 \mathcal{R} -rigidity

We are going to prove that Liouville billiard tables of classical type are \mathcal{R} -rigid with respect to the densities μ defined by (1.1).

Theorem 4.1. *Let (X, \tilde{g}) be a Liouville billiard table of classical type and let $K \in C(\Gamma, \mathbb{R})$ be invariant with respect to the action of the group $G(X)$ on Γ . Suppose that $\mathcal{R}_{K,\mu}(\Lambda) = 0$ for any Liouville torus $\Lambda \subset \mathcal{F}_b$. Then $K \equiv 0$.*

Proof of Theorem 4.1. First, consider the case when $\mu \equiv 1$. Denote the pull-back of K under the projection $\sigma|_{\mathbb{T}_N^2} : \mathbb{T}_N^2 \rightarrow \Gamma$ by \mathcal{K} , $\mathcal{K} \in C(\mathbb{T}_N^2)$. Let $\Lambda_h \in \mathcal{F}_b$ be a Liouville torus and let T_h be a connected component of $(\sigma|_{\mathbb{T}_N^2})^* \Lambda_h \subset B^*(\mathbb{T}_N^2)$ where $h = (h_1, h_2)$ are the values of the integrals \mathcal{I}_1 and \mathcal{I}_2 on T_h . Then we have

$$(\lambda_h(T_h))^{-1} \int_{T_h} \mathcal{K} \lambda_h = 2\mathcal{R}_{K,1}(\Lambda_h) = 0,$$

where λ_h is the corresponding Leray's form on T_h . Note that \mathcal{K} is invariant under the involution $(\theta_1, \theta_2) \mapsto (-\theta_1, \frac{\omega_2}{2} - \theta_2)$ since $\sigma|_{\mathbb{T}_N^2}$ is invariant under the involution (3.1) for $\theta_3 = N$. Recall that that the group $G(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ defined by (3.18) acts by isometries on X and on its boundary Γ . Since K is invariant under this action, the function $\mathcal{K}(\theta_1, \theta_2)$ is invariant with respect to the involutions

$$(\theta_1, \theta_2) \mapsto (-\theta_1, \theta_2), \quad (\theta_1, \theta_2) \mapsto (\omega_1/2 - \theta_1, \theta_2) \quad (4.1)$$

and

$$(\theta_1, \theta_2) \mapsto (\theta_1, -\theta_2), \quad (\theta_1, \theta_2) \mapsto (\theta_1, \omega_2/2 - \theta_2). \quad (4.2)$$

From now on we consider $\mathcal{K} \in C(\mathbb{T}_N^2, \mathbb{R})$ which is invariant with respect to the involutions (4.1) and (4.2) and such that for any

$$\int_{T_h} \mathcal{K} \lambda_h = 0 \quad \forall T_h \in \mathcal{F}_b. \quad (4.3)$$

First, take $h = (h_1, h_2)$ and assume, for example, that $T_h \subset \tilde{U}_1$. We shall give an explicit formula for the Leray form on the connected components of \tilde{L}_h , using the parameterization obtained in Sect. 3.3. Set $T_h := T_h^{(1)}$, and let T_h^+ be the “half torus” $r_{11}(A'_h)$, where the map r_{11} is defined by (3.31). Consider the set

$$A'_h(\delta) := \{0 \leq \theta_1 \leq \omega_1; -f(\kappa_2) + \delta \leq \theta_2 \leq f(\kappa_2) - \delta\},$$

where $\delta > 0$ is sufficiently small. It follows from (3.31) that the functions $(\theta_1, \theta_2, \mathcal{I}_1, \mathcal{I}_2)$ give a coordinate chart in a neighborhood of the branch $T_h^+(\delta) := r_{11}(A'_h(\delta)) \subseteq T_h^+$. We will compute the Leray form on it. In the coordinates $\{(\theta_1, \theta_2; p_1, p_2)\}$ on $B^*\mathbb{T}_N^2$ we have

$$\begin{aligned} \tilde{\omega}_2 \wedge \tilde{\omega}_2 &= 2 dp_1 \wedge d\theta_1 \wedge dp_2 \wedge d\theta_2 \\ &= 2 d(\sqrt{\varphi_1(\theta_1)^2 - \mathcal{I}_1 \varphi_1(\theta_1) + \mathcal{I}_2}) \wedge d\theta_1 \wedge d(\sqrt{-\varphi_2(\theta_2)^2 + \mathcal{I}_2 \varphi_2(\theta_2) - \mathcal{I}_1}) \wedge d\theta_2 \\ &= -\frac{1}{2} \frac{(\varphi_1(\theta_1) - \varphi_2(\theta_2)) d\theta_1 \wedge d\theta_2}{\sqrt{\varphi_1(\theta_1)^2 - \mathcal{I}_1 \varphi_1(\theta_1) + \mathcal{I}_2} \sqrt{-\varphi_2(\theta_2)^2 + \mathcal{I}_2 \varphi_2(\theta_2) - \mathcal{I}_1}} \wedge d\mathcal{I}_1 \wedge d\mathcal{I}_2. \end{aligned}$$

In particular, letting $\delta \rightarrow 0+0$ we see that the Leray form on T_h^+ can be identified with

$$\lambda_h := \frac{(\varphi_1(\theta_1) - \varphi_2(\theta_2)) d\theta_1 \wedge d\theta_2}{\sqrt{\varphi_1(\theta_1)^2 - h_1 \varphi_1(\theta_1) + h_2} \sqrt{-\varphi_2(\theta_2)^2 + h_1 \varphi_2(\theta_2) - h_2}}. \quad (4.4)$$

We have

$$\begin{aligned}\int_{T_h} \mathcal{K} \lambda_h &= 2 \int_{-f_2(\kappa_2)}^{f_2(\kappa_2)} \int_0^{\omega_1} \frac{\mathcal{K}(\theta_1, \theta_2)(\varphi_1(\theta_1) - \varphi_2(\theta_2)) d\theta_1 d\theta_2}{\sqrt{(\varphi_1(\theta_1) - \kappa_1)(\varphi_1(\theta_1) - \kappa_2)} \sqrt{(\varphi_2(\theta_2) - \kappa_1)(\kappa_2 - \varphi_2(\theta_2))}} \\ &= 16 \int_0^{f_2(\kappa_2)} \int_0^{\omega_1/4} \frac{\mathcal{K}(\theta_1, \theta_2)(\varphi_1(\theta_1) - \varphi_2(\theta_2)) d\theta_1 d\theta_2}{\sqrt{(\varphi_1(\theta_1) - \kappa_1)(\varphi_1(\theta_1) - \kappa_2)} \sqrt{(\varphi_2(\theta_2) - \kappa_1)(\kappa_2 - \varphi_2(\theta_2))}},\end{aligned}$$

as the functions \mathcal{K} , φ_1 , and φ_2 are invariant with respect to the involutions (4.1) and (4.2). Set $\tilde{K}(\theta_1, \theta_2) := \mathcal{K}(\theta_1, \theta_2)(\varphi_1(\theta_1) - \varphi_2(\theta_2))$ and denote

$$M_A(\kappa_1, \kappa_2) := \int_0^{f_2(\kappa_2)} \int_0^{\omega_1/4} \frac{\tilde{K}(\theta_1, \theta_2) d\theta_1 d\theta_2}{\sqrt{(\varphi_1(\theta_1) - \kappa_1)(\varphi_1(\theta_1) - \kappa_2)} \sqrt{(\varphi_2(\theta_2) - \kappa_1)(\kappa_2 - \varphi_2(\theta_2))}}. \quad (4.5)$$

Then (4.3) implies

$$M_A(\kappa_1, \kappa_2) = 0$$

for any $\kappa_1 \in (-\nu_3, 0)$ and any $\kappa_2 \in (0, \nu_1)$.

Remark 4.2. Note that for any fixed $\kappa_2 \in (0, \nu_1)$ the function $\kappa_1 \rightarrow M_A(\kappa_1, \kappa_2)$ can be extended to an analytic (possibly multivalued) function on $\mathbb{C} \setminus ([0, \kappa_2] \sqcup [\nu_1, \nu_0])$. Since it vanishes for $\kappa_1 \in (-\nu_3, 0)$ we obtain that $M_A(\kappa_2, \kappa_1) \equiv 0$, $\forall \kappa_1 \in \mathbb{C} \setminus ([0, \kappa_2] \sqcup [\nu_1, \nu_0])$ and $\forall \kappa_2 \in (0, \nu_1)$.

Set

$$\tilde{K}_1(x_1, x_2) := \tilde{K}(f_1(x_1), f_2(x_2)) f'_1(x_1) f'_2(x_2).$$

It follows from (3.27) and (3.29) that $\tilde{K}_1 \in L^1((\nu_1, \nu_0) \times (0, \nu_1))$. More precisely, (3.27) and (3.29) imply

Lemma 4.3. We have

$$\tilde{K}_1(x_1, x_2) = \frac{\tilde{K}(f_1(x_1), f_2(x_2)) F(x_1, x_2)}{\sqrt{x_1 - \nu_1} \sqrt{\nu_0 - x_1} \sqrt{x_2} \sqrt{\nu_1 - x_2}}, \quad (4.6)$$

where the function $(x_1, x_2) \mapsto K(f_1(x_1), f_2(x_2))$ is continuous on $[\nu_1, \nu_0] \times [0, \nu_1]$, the function $F \in C([\nu_1, \nu_0] \times [0, \nu_1])$ does not dependent on \tilde{K} and $F > 0$.

Passing to the variables $x_1 = \varphi_1(\theta_1)$ and $x_2 = \varphi_2(\theta_2)$ in (4.5) we get

$$M_A(\kappa_1, \kappa_2) = \int_{\nu_1}^{\nu_0} \int_0^{\kappa_2} \frac{\tilde{K}_1(x_1, x_2) dx_2 dx_1}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)} \sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \equiv 0 \quad (4.7)$$

for any $\kappa_1 \in (-\infty, 0)$ and any $\kappa_2 \in (0, \nu_1)$. Consider now the case (B). Arguing in the same way we obtain

$$M_B(\kappa_1, \kappa_2) := \frac{1}{16} \int_{T_h} \mathcal{K} \lambda_h = \int_0^{\nu_1} \int_{\kappa_2}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2) dx_1 dx_2}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)} \sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \equiv 0 \quad (4.8)$$

for any $\kappa_1 \in (-\infty, 0)$ and $\kappa_2 \in (\nu_1, \nu_0)$.

In the same way one obtains:

CASE (C): For any $0 < \kappa_1 < \kappa_2 < \nu_1$,

$$M_C(\kappa_1, \kappa_2) := \int_{T_h} \mathcal{K} \lambda_h = \int_{\kappa_1}^{\kappa_2} \int_{\nu_1}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2) dx_1 dx_2}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)} \sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}}. \quad (4.9)$$

CASE (D): For any $\kappa_1 \in (0, \nu_1)$ and $\kappa_2 \in (\nu_1, \nu_0)$,

$$M_D(\kappa_1, \kappa_2) := \int_{T_h} \mathcal{K} \lambda_h = \int_{\kappa_1}^{\nu_1} \int_{\kappa_2}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2) dx_1 dx_2}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)} \sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}}. \quad (4.10)$$

Remark 4.4. In what follows we will not use the identities (4.9) and (4.10).

Now, we argue as follows: Take a continuous function χ on the interval $[0, \nu_1]$ and consider the mean

$$\overline{M}_A(\varphi; \kappa_1) := \int_0^{\nu_1} M_A(\kappa_1, \kappa_2) \chi(\kappa_2) d\kappa_2,$$

where $M_A(\kappa_1, \kappa_2)$ is given by (4.7). In view of Lemma 4.3, we can apply Fubini's theorem to the following integral

$$\begin{aligned} 0 &\equiv \overline{M}_A(\varphi; \kappa_1) = \int_0^{\nu_1} \left(\int_0^{\kappa_2} \int_{\nu_1}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2) \chi(\kappa_2) dx_1 dx_2}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)} \sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \right) d\kappa_2 \\ &= \int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2)}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)}} \left[\int_{x_2}^{\nu_1} \frac{\chi(\kappa_2) d\kappa_2}{\sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \right] dx_1 dx_2 \end{aligned} \quad (4.11)$$

for any $\kappa_1 \in (-\infty, 0)$ and any $\chi \in C([0, \nu_1])$. Similarly, consider the mean

$$\overline{M}_B(\varphi; \kappa_1) := \int_{\nu_1}^{\nu_0} M_B(\kappa_1, \kappa_2) \chi(\kappa_2) d\kappa_2$$

where χ is a continuous function on the interval $[\nu_1, \nu_0]$. We obtain as above

$$\begin{aligned} 0 &\equiv \overline{M}_B(\varphi; \kappa_1) = \int_{\nu_1}^{\nu_0} \int_0^{\nu_1} \int_{\kappa_2}^{\nu_1} \frac{\tilde{K}_1(x_1, x_2) \chi(\kappa_2) dx_1 dx_2 d\kappa_2}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)} \sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \\ &= \int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2)}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)}} \left[\int_{\nu_1}^{x_1} \frac{\chi(\kappa_2) d\kappa_2}{\sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \right] dx_1 dx_2 \end{aligned} \quad (4.12)$$

for any $\kappa_1 \in (-\infty, 0)$. Finally, combining (4.11) and (4.12) we obtain for any $\chi \in C([0, \nu_0])$ and any $\kappa_1 \in (-\infty, 0)$ the equality

$$\int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2)}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)}} \left[\int_{x_2}^{x_1} \frac{\chi(\kappa_2) d\kappa_2}{\sqrt{(x_1 - \kappa_2)(\kappa_2 - x_2)}} \right] dx_1 dx_2 \equiv 0. \quad (4.13)$$

In particular, for any $k \geq 0$ and for any $\kappa_1 \in (-\infty, 0)$,

$$\int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \frac{\tilde{K}_1(x_1, x_2)}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)}} R_k(x_1, x_2) dx_1 dx_2 \equiv 0, \quad (4.14)$$

where

$$R_k(x_1, x_2) := \int_{x_2}^{x_1} \frac{z^k dz}{\sqrt{(x_1 - z)(z - x_2)}} = \int_0^1 \frac{(x_2 + t(x_1 - x_2))^k}{\sqrt{t(1-t)}} dt. \quad (4.15)$$

Recall that the Legendre polynomials P_k , $k \geq 0$, can be generated by the power series expansion,

$$(1 - 2\omega z + z^2)^{-1/2} = \sum_{k=0}^{\infty} P_k(\omega) z^k, \quad (4.16)$$

which is convergent for small z . For $0 < x_2 \leq x_1$ we set $s_1 := (x_1 + x_2)/2$ and $s_2 := \sqrt{x_1 x_2}$.

Lemma 4.5. *For any $k \geq 0$ and for any $0 < x_2 \leq x_1$, $R_k(x_1, x_2) = \pi s_2^k P_k(s_1/s_2)$.*

Proof. For any given values of x_1 and x_2 , $0 < x_2 \leq x_1$, consider the power series in z ,

$$I(z) := \sum_{k=0}^{\infty} R_k(x_1, x_2) z^k$$

There exists $0 < r < \infty$ sufficiently small such that the power series converges for $|z| \leq r$ and

$$I(z) = \int_0^1 \frac{1}{t} \frac{1}{1 - z(x_2 + t(x_1 - x_2))} \sqrt{\frac{t}{1-t}} dt.$$

Using the substitution, $s = \sqrt{\frac{t}{1-t}}$ we get

$$I(z) = \frac{\pi}{\sqrt{(1 - zx_1)(1 - zx_2)}},$$

and by (4.16) we obtain

$$I(z) = \pi \sum_{k=0}^{\infty} s_2^k P_k(s_1/s_2) z^k,$$

which proves the lemma. \square

Note that the function

$$(x_1, x_2) \mapsto Q(x_1, x_2, \kappa_1) := \frac{\tilde{K}_1(x_1, x_2)}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)}}$$

belongs to $L^1([\nu_1, \nu_0] \times [0, \nu_1])$ in view of Lemma 4.3, and it depends analytically on $\kappa_1 \in (-\infty, 0)$. Consider the power series expansion

$$\frac{1}{\sqrt{(x_1 - \kappa_1)(x_2 - \kappa_1)}} = \sum_{j=1}^{\infty} Q_j(x_1, x_2) \kappa_1^{-j}, \quad (4.17)$$

where $(x_1, x_2) \in [\nu_1, \nu_0] \times [0, \nu_1]$ and $\kappa_1 < 0$. Now (4.16) implies $Q_j(x_1, x_2) = -s_2^{j-1} P_{j-1}(s_1/s_2)$ for any $j \geq 1$.

Using Lemma 4.5, (4.14) and (4.17) we obtain that for any $k, j \geq 0$,

$$\int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \tilde{K}_1(x_1, x_2) s_2^{k+j} P_k(s_1/s_2) P_j(s_1/s_2) dx_1 dx_2 = 0. \quad (4.18)$$

Let k and m be non-negative integers such that $2k \leq m$ and let d be the integer part of $m/2$. We have the following relation due to Adams (see [1], [16, Chap. XV, Legendre functions, Miscellaneous Examples, Ex. 11]),

$$P_k(z)P_{m-k}(z) = \sum_{r=0}^k c_{k,r}^m P_{m-2r}(z) = \sum_{r=0}^d c_{k,r}^m P_{m-2r}(z),$$

where for any $0 \leq r \leq d$,

$$c_{k,r}^m := \frac{A_{k-r} A_r A_{m-k-r}}{A_{m-r}} \cdot \frac{2m - 4r + 1}{2m - 2r + 1},$$

with

$$A_k := \begin{cases} \frac{1 \cdot 3 \cdot 5 \dots (2k-1)}{k}, & k \geq 1, \\ 1, & k = 0, \\ A_k = 0, & k \leq -1. \end{cases}$$

Hence, for any given $m \geq 0$ we obtain a $(d+1) \times (d+1)$ matrix $(c_{k,r}^m)_{k,r=0}^d$ which is triangular (all the elements over the diagonal vanish) and with non-vanishing diagonal elements. This together with (4.18) (take $m = j+k$) implies that for any $m \geq 0$, $0 \leq 2r \leq m$,

$$\int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \tilde{K}_1(x_1, x_2) s_2^m P_{m-2r}(s_1/s_2) dx_1 dx_2 = 0.$$

On the other hand, for any $m \geq 0$ the monomial z^m can be written as a linear combination of the Legendre polynomials $P_{m-2r}(z)$, $0 \leq 2r \leq m$, and we get

$$\int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \tilde{K}_1(x_1, x_2) s_1^{m-2r} s_2^{2r} dx_1 dx_2 = 0. \quad (4.19)$$

Consider the set of monomials $\mathcal{M} = \{s_1^{m-2r} s_2^{2r} : r, m \in \mathbb{Z}, 0 \leq 2r \leq m\}$. Obviously \mathcal{M} is closed under multiplication, $1 \in \mathcal{M}$, and it separates the points (x_1, x_2) of the compact $[\nu_1, \nu_0] \times [0, \nu_1]$, since $s_1, s_2^2 \in \mathcal{M}$ and $0 \leq x_2 \leq x_1$ are the unique solutions of $x^2 - 2s_1x + s_2^2 = 0$. The Stone-Weierstrass theorem implies that the vector space $\text{Span}(\mathcal{M})$ of all finite linear combinations of monomials of \mathcal{M} is dense in $C([\nu_1, \nu_0] \times [0, \nu_1])$. Choose $\psi \in C([\nu_1, \nu_0] \times [0, \nu_1])$. Then for any $\varepsilon > 0$ there is $P \in \text{Span}(\mathcal{M})$ such that

$$\|P - \psi\|_{C([\nu_1, \nu_0] \times [0, \nu_1])} < \varepsilon.$$

Now (4.19) implies

$$\begin{aligned} & \left| \int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \tilde{K}_1(x_1, x_2) \psi(x_1, x_2) dx_1 dx_2 \right| \\ & \leq \int_0^{\nu_1} \int_{\nu_1}^{\nu_0} |\tilde{K}_1(x_1, x_2)| |\psi(x_1, x_2) - P(x_1, x_2)| dx_1 dx_2 \leq \varepsilon \|\tilde{K}_1\|_{L^1((\nu_1, \nu_0) \times (0, \nu_1))}. \end{aligned}$$

Hence,

$$\int_0^{\nu_1} \int_{\nu_1}^{\nu_0} \tilde{K}_1(x_1, x_2) \psi(x_1, x_2) dx_1 dx_2 = 0$$

for any $\psi \in C([\nu_1, \nu_0] \times [0, \nu_1])$ which implies $\tilde{K}_1 \equiv 0$ on that compact. In particular, $\mathcal{K} \equiv 0$, and hence $K \equiv 0$. This completes the proof when $\mu \equiv 1$.

Now, consider the case when $\mu(\xi) = \langle \pi^+(\xi), n_g \rangle^{-1}$. Assume that $\xi \in \Lambda_h$ where Λ_h is a Liouville torus in \mathcal{F}_b and $h = (h_1, h_2)$ are the values of the integrals \tilde{I}_1 and \tilde{I}_2 on Λ_h . Let $\Lambda_h \subset U_1$. Using the mapping (3.31), we introduce coordinates $\{(\theta_1, \theta_2)\}$ on the “half” tori $T_{h+}^{(1)} = r_{11}(A'_h)$ and $T_{h-}^{(1)} = r_{1,-1}(A'_h)$ of $T_h^{(1)}$ as well as on $T_{h+}^{(2)} = r_{-11}(A'_h)$ and $T_{h-}^{(2)} = r_{-1,-1}(A'_h)$ of $T_h^{(2)}$. Similarly, we parametrize the Liouville tori Λ_h in U_2 .

Lemma 4.6. *In coordinates $\{(\theta_1, \theta_2)\}$ on $\Lambda_h \subset \mathcal{F}_b$, $\mu(\xi) = \langle \pi^+(\xi), n_g \rangle^{-1}$ is given by*

$$\mu(\theta_1, \theta_2) = \frac{\sqrt{(\varphi_1(\theta_1) - \nu_3)(\varphi_1(\theta_1) - \nu_3)}}{\sqrt{(\kappa_1 - \nu_3)(\kappa_2 - \nu_3)}} \quad (4.20)$$

where $\kappa_1 = h_1 + h_2$ and $\kappa_2 = h_1 h_2$.

Proof of Lemma 4.6. Fix $h = (h_1, h_2)$ so that $\Lambda_h \subset \mathcal{F}_b$. It follows from (3.5) that

$$n_g = -\frac{1}{\sqrt{\Pi_3}} \frac{\partial}{\partial \theta_3}.$$

On the other hand, the third equation in (3.13) shows that

$$p_3 = -\sqrt{\nu_3^2 - h_1 \nu_3 + h_2} = -\sqrt{(\kappa_1 - \nu_3)(\kappa_2 - \nu_3)}.$$

Hence, $\langle \pi^+(\xi), n_g \rangle = \sqrt{(\kappa_1 - \nu_3)(\kappa_2 - \nu_3)} / \sqrt{\Pi_3}$. \square

Remark 4.7. *The statement of Lemma 4.6 holds also for any $\Lambda_h \in \mathcal{F}$ not necessarily in \mathcal{F}_b .*

Note that the denominator in (4.20) is a positive constant on T_h and the numerator is independent of h_1 and h_2 and does not vanish. The relation (4.3) with

$$\tilde{K}(\theta_1, \theta_2) = \mathcal{K}(\theta_1, \theta_1)(\varphi_1(\theta_1) - \varphi_2(\theta_2))\sqrt{(\varphi_1(\theta_1) - \nu_3)(\varphi_1(\theta_1) - \nu_3)}.$$

implies that the expression (4.5) vanishes. In particular, (4.7) and (4.8) hold. Finally, arguing in the same way as in the case $\mu \equiv 1$ one concludes that $K \equiv 0$. \square

5 Non-degeneracy of the frequency map

In this section we investigate the non-degeneracy of the frequency map of Liouville billiard tables of classical type.

Theorem 5.1. *Let (X, g) be an analytic 3-dimensional Liouville billiard table of classical type. Suppose that there is at least one non-periodic geodesic on Γ . Then the frequency map is non-degenerate in the union $U_1 \cup U_2$ corresponding to the boundary cases (A) and (B).*

Proof. As in Sect. 3.3 we introduce coordinates $\{(\theta_1, \theta_2, \theta_3; p_1, p_2, p_3)\}$ on the cotangent bundle T^*C , where p_1, p_2, p_3 are the conjugate variables to θ_1, θ_2 , and θ_3 . Solving the system of equations (3.12) with respect to p_1^2, p_2^2 , and p_3^2 , where $H = 1$, $I_1 = h_1$, and $I_2 = h_2$ are given values of the integrals, we get

$$\begin{cases} p_1^2 &= \varphi_1^2 - h_1\varphi_1 + h_2 \\ p_2^2 &= -(\varphi_2^2 - h_1\varphi_2 + h_2) \\ p_3^2 &= \varphi_3^2 - h_1\varphi_3 + h_2. \end{cases} \quad (5.1)$$

In particular, it follows from (5.1) that the invariant set

$$\mathcal{T}_h := \{H = 1, I_1 = h_1, I_2 = h_2\} \subset T^*C \quad (5.2)$$

is non-empty if and only if the quadratic polynomial $P(t) = t^2 - h_1t + h^2$ has real roots $\kappa_1 \leq \kappa_2$ (i.e., $\mathcal{D} = h_1^2 - 4h_2 \geq 0$). As in Sect. 4 we obtain four cases related to the position of the roots κ_1 and κ_2 with respect to the constants $\nu_3 < \nu_2 = 0 < \nu_1 < \nu_0$, namely,

- (A) $\nu_3 \leq \kappa_1 \leq \nu_2 = 0$ and $0 \leq \kappa_2 \leq \nu_1$;
- (B) $\nu_3 \leq \kappa_1 \leq 0$ and $\nu_1 \leq \kappa_2 \leq \nu_0$;
- (C) $0 \leq \kappa_1 \leq \kappa_2 \leq \nu_1$;
- (D) $0 \leq \kappa_1 \leq \nu_1$ and $\nu_1 \leq \kappa_2 \leq \nu_0$.

Recall that $\nu_3 = \min \varphi_3$, $\nu_2 = \max \varphi_3 = \min \varphi_2 = 0$, $\nu_1 = \max \varphi_2 = \min \varphi_1$, and $\nu_0 = \max \varphi_1$. In what follows we consider κ_1 and κ_2 as new parameters (constants of motion)⁵ that parametrize the invariant set (5.2).

We first consider the case (A) where $\nu_3 \leq \kappa_1 \leq \nu_2 = 0$ and $0 = \nu_2 \leq \kappa_2 \leq \nu_1$. It follows from (5.1) that the impulses are real-valued if and only if

$$\begin{cases} \nu_1 \leq \varphi_1(\theta_1) \leq \nu_0 \\ 0 \leq \varphi_2(\theta_2) \leq \kappa_2 \\ \nu_3 \leq \varphi_3(\theta_3) \leq \kappa_1 \end{cases} \quad (5.3)$$

Hence, the projection of the invariant set (5.2) onto the base C is described by the following inequalities:

$$\begin{aligned} 0 &\leq \theta_1 \leq \omega_1; \\ -f_2(\kappa_2) &\leq \theta_2 \leq f_2(\kappa_2) \quad \text{or} \quad -f_2(\kappa_2) + \frac{\omega_2}{2} \leq \theta_2 \leq f_2(\kappa_2) + \frac{\omega_2}{2}; \\ f_3(\kappa_1) &\leq \theta_3 \leq N \quad \text{or} \quad -N \leq \theta_3 \leq -f_3(\kappa_1), \end{aligned}$$

where f_2 is the inverse of $\varphi_2|_{[0, \omega_2/4]}$ and f_3 is the inverse of $\varphi_3|_{[0, N]}$. These inequalities give four rectangular boxes in C that project onto an unique set in \tilde{C} via the projection (3.3). Consider,

⁵ $h_1 = \kappa_1 + \kappa_2$ and $h_2 = \kappa_1\kappa_2$

for example, the rectangular box B_h given by

$$B_h : \begin{cases} 0 \leq \theta_1 \leq \omega_1; \\ -f_2(\kappa_2) \leq \theta_2 \leq f_2(\kappa_2); \\ f_3(\kappa_1) \leq \theta_3 \leq N. \end{cases} \quad (5.4)$$

For any given $\theta \in B_h$ we obtain from (5.1) that

$$\begin{aligned} p_1(\theta) &= \epsilon_1 \sqrt{(\varphi_1(\theta_1) - \kappa_1)(\varphi_1(\theta_1) - \kappa_2)} \\ p_2(\theta) &= \epsilon_2 \sqrt{(\varphi_2(\theta_2) - \kappa_1)(\kappa_2 - \varphi_2(\theta_2))} \\ p_3(\theta) &= \epsilon_3 \sqrt{(\kappa_1 - \varphi_3(\theta_3))(\kappa_2 - \varphi_3(\theta_3))} \end{aligned}$$

where $\epsilon_k = \pm 1$. Then the mapping $r_+ : B_h \rightarrow T^*C$,

$$(\theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \theta_2, \theta_3; p_1(\theta), p_2(\theta), p_3(\theta)),$$

where $\epsilon_1 = 1$, $\epsilon_2 = \pm 1$, and $\epsilon_3 = \pm 1$, parametrizes one of the two connected components of the subset $T_h := \{\tilde{H} = 1, \tilde{I}_1 = h_1, \tilde{I}_2 = h_2\} \subset T^*X$. Assume that the strict inequalities $\nu_3 < \kappa_1 < \nu_2 = 0$ and $0 < \kappa_2 < \nu_1$ hold.

Remark 5.2. *This component is diffeomorphic to $\mathbb{T}^2 \times [0, 1]$ and its intersection with the boundary $T^*X|_\Gamma$ of T^*X has two components which can be identified with the two components of the image of the slice*

$$\{0 \leq \theta_1 \leq \omega_1, -f_2(\kappa_2) \leq \theta_2 \leq f_2(\kappa_2), \theta_3 = N\}$$

of B_h with respect to r_+ with $\epsilon_3 = 1$ and $\epsilon_3 = -1$ respectively. In particular, the impulse p_3 takes constant values of different sign on them. Moreover, the reflection map $r : T^*X|_\Gamma \rightarrow T^*X|_\Gamma$ is given by

$$(\theta_1, \theta_2; p_1, p_2, p_3) \mapsto (\theta_1, \theta_2; p_1, p_2, -p_3).$$

Hence, the reflection map interchanges these two components, and by Lemma 7.4 (c), $m = 1$ (cf. Remark 2.1). Similarly, we get $m = 1$ in the case (B).

Now we compute the generalized actions of the billiard flow corresponding to T_h (see (7.8), Appendix),

$$\begin{aligned} J_1(\kappa_1, \kappa_2) &= \frac{1}{2\pi} \int_0^{\omega_1} \sqrt{(\varphi_1(\theta_1) - \kappa_1)(\varphi_1(\theta_1) - \kappa_2)} d\theta_1 \\ &= \frac{2}{\pi} \int_{\nu_1}^{\nu_0} \sqrt{(x_1 - \kappa_1)(x_1 - \kappa_2)} \rho_1(x_1) dx_1, \end{aligned} \quad (5.5)$$

$$\begin{aligned} J_2(\kappa_1, \kappa_2) &= \frac{1}{\pi} \int_{-f_2(\kappa_2)}^{f_2(\kappa_2)} \sqrt{(\varphi_2(\theta_2) - \kappa_1)(\kappa_2 - \varphi_2(\theta_2))} d\theta_2 \\ &= \frac{2}{\pi} \int_0^{\kappa_2} \sqrt{(x_2 - \kappa_1)(\kappa_2 - x_2)} \rho_2(x_2) dx_2, \end{aligned} \quad (5.6)$$

$$\begin{aligned}
J_3(\kappa_1, \kappa_2) &= \frac{1}{\pi} \int_{f_3(\kappa_1)}^N \sqrt{(\kappa_1 - \varphi_3(\theta_3))(\kappa_2 - \varphi_3(\theta_3))} d\theta_3 \\
&= \frac{1}{\pi} \int_{\nu_3}^{\kappa_1} \sqrt{(\kappa_1 - x_3)(\kappa_2 - x_3)} \rho_3(x_3) dx_3,
\end{aligned} \tag{5.7}$$

where

$$\rho_1(x_1) := f'_1(x_1) > 0, \quad \rho_2(x_2) := f'_2(x_2) > 0 \quad \text{and} \quad \rho_3(x_3) := -f'_3(x_3) > 0$$

are analytic functions in the intervals (ν_1, ν_0) , $(0, \nu_1)$ and $(\nu_3, 0)$, respectively, and f_1 and f_2 satisfy (3.27) and (3.29). Notice that the functions F_1^\pm and F_2^\pm in (3.27) and (3.29) are smooth and even analytic in a neighborhood of 0, and $f_3(x_3) = \sqrt{-x_3} F_3(\sqrt{-x_3})$, where F_3 is analytic in a neighborhood of $[\nu_3, 0]$. By the assumption (A_5) , $\rho_3(x_3)$ is smooth at $x_3 = \nu_3$. In particular, we obtain

Remark 5.3. *The functions ρ_1 , ρ_2 and ρ_3 are analytic in the intervals (ν_1, ν_0) , $(0, \nu_1)$ and $(\nu_3, 0)$, respectively, and*

$$\begin{aligned}
\rho_1(x_1) &= \frac{G_1^-(\sqrt{x_1 - \nu_1})}{\sqrt{x_1 - \nu_1}} \quad \text{as } x_1 \rightarrow \nu_1 + 0 \quad \text{and} \quad \rho_1(x_1) = \frac{G_1^+(\sqrt{\nu_0 - x_1})}{\sqrt{\nu_0 - x_1}} \quad \text{as } x_1 \rightarrow \nu_0 - 0 \\
\rho_2(x_2) &= \frac{G_2^-(\sqrt{x_2})}{\sqrt{x_2}} \quad \text{as } x_2 \rightarrow 0 + 0 \quad \text{and} \quad \rho_2(x_2) = \frac{G_2^+(\sqrt{\nu_1 - x_2})}{\sqrt{\nu_1 - x_2}} \quad \text{as } x_2 \rightarrow \nu_1 - 0 \quad \text{and} \\
\rho_3(x_3) &= \frac{G_3(\sqrt{-x_3})}{\sqrt{-x_3}} \quad \text{as } x_3 \rightarrow 0 - 0,
\end{aligned}$$

where G_1^\pm and G_2^\pm are analytic in a neighborhood of 0, and G_3 is analytic in a neighborhood of $[\nu_3, 0]$. Moreover, $G_3(0) > 0$ and by (3.28) and (3.30) we have

$$\begin{aligned}
G_1^+(0) &= \sqrt{2\varphi_1''(0)^{-1}}, \quad G_1^-(0) = -\sqrt{-2\varphi_1''(\omega_1/4)^{-1}}, \\
G_2^+(0) &= \sqrt{2\varphi_2''(0)^{-1}}, \quad G_2^-(0) = -\sqrt{-2\varphi_2''(\omega_2/4)^{-1}}.
\end{aligned}$$

As a corollary we obtain

Lemma 5.4. *The functions J_1 , J_2 , and J_3 are analytic in $(\kappa_1, \kappa_2) \in (\nu_3, 0) \times (0, \nu_1)$.*

Proof of Lemma 5.4. The function J_1 is obviously analytic in that domain. Fix $a \in (0, \nu_1)$ and take $0 < \delta \ll 1$ such that $\rho_2(z)$ is holomorphic in the disc $\mathbb{D}_{2\delta}(a) := \{|z - a| < 2\delta\} \subset \mathbb{C}$. Then write

$$J_2(\kappa_1, \kappa_2) = \int_0^{a-\delta} \sqrt{\kappa_2 - x_2} f(x_2, \kappa_1) dx_2 + \int_{a-\delta}^{\kappa_2} \sqrt{\kappa_2 - x_2} f(x_2, \kappa_1) dx_2,$$

where

$$f(x_2, \kappa_1) = \frac{1}{\pi} \sqrt{x_2 - \kappa_1} \rho_2(x_2)$$

is analytic in $(0, \nu_1) \times (\nu_3, 0)$. Then the first integral defines an analytic function in $(\kappa_1, \kappa_2) \in (\nu_3, 0) \times (a - \delta/2, a + \delta/2)$. Consider now the second one. We expand $f(x_2, \kappa_1)$ in Taylor series with respect to x_2 at $x_2 = \kappa_2$. Then integrating with respect to x_2 and using Cauchy inequalities for $\frac{d^j f}{dx_2^j}(\kappa_2, \kappa_1)$, where $(\kappa_2, \kappa_1) \in \mathbb{D}_{\delta/2}(a) \times (\nu_3 + \delta, -\delta)$, we obtain that the second integral defines an analytic function in $(\kappa_1, \kappa_2) \in (\nu_3 + \delta, -\delta) \times \mathbb{D}_{\delta/2}(a)$. In the same way we prove that J_3 is analytic in $(\nu_3, 0) \times (0, \nu_1)$. \square

In order to obtain suitable formulas for the frequencies of the billiard ball map we proceed as in the Appendix. Denote by $\mathcal{H}(J_1, J_1, J_3)$ the Hamiltonian of the billiard flow expressed in the corresponding action-angle coordinates. Then for any κ_1 and κ_2 such that $\nu_3 < \kappa_1 < 0$ and $0 < \kappa_2 < \nu_1$, one has

$$\mathcal{H}(J_1(\kappa_1, \kappa_2), J_1(\kappa_1, \kappa_2), J_3(\kappa_1, \kappa_2)) \equiv 1. \quad (5.8)$$

Differentiating (5.8) with respect to κ_1 and κ_2 we get that the frequencies Ω_1 and Ω_2 of the billiard ball map satisfy

$$\begin{bmatrix} \frac{\partial J_1}{\partial \kappa_1} & \frac{\partial J_2}{\partial \kappa_1} & \frac{\partial J_3}{\partial \kappa_1} \\ \frac{\partial J_1}{\partial \kappa_2} & \frac{\partial J_2}{\partial \kappa_2} & \frac{\partial J_3}{\partial \kappa_2} \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ 2\pi \end{bmatrix} = 0$$

and therefore (cf. formula (7.7) in the Appendix)

$$\begin{bmatrix} \frac{\partial J_1}{\partial \kappa_1} & \frac{\partial J_2}{\partial \kappa_1} \\ \frac{\partial J_1}{\partial \kappa_2} & \frac{\partial J_2}{\partial \kappa_2} \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix} = -2\pi \begin{bmatrix} \frac{\partial J_3}{\partial \kappa_1} \\ \frac{\partial J_3}{\partial \kappa_2} \end{bmatrix}. \quad (5.9)$$

The latter relation and the formulas for the actions (5.5)-(5.7) lead to the following formulas for the frequencies

$$\Omega_1(\kappa_1, \kappa_2) = \pi \frac{A(\kappa_1, \kappa_2)}{D(\kappa_1, \kappa_2)} \quad \text{and} \quad \Omega_2(\kappa_1, \kappa_2) = \pi \frac{B(\kappa_1, \kappa_2)}{D(\kappa_1, \kappa_2)}, \quad (5.10)$$

where

$$A(\kappa_1, \kappa_2) := \int_{\nu_3}^{\kappa_1} \int_0^{\kappa_2} \frac{(x_2 - x_3)\rho_2(x_2)\rho_3(x_3) dx_2 dx_3}{\sqrt{(x_2 - \kappa_1)(\kappa_2 - x_2)(\kappa_1 - x_3)(\kappa_2 - x_3)}},$$

$$B(\kappa_1, \kappa_2) := \int_{\nu_3}^{\kappa_1} \int_{\nu_1}^{\nu_0} \frac{(x_1 - x_3)\rho_1(x_1)\rho_3(x_3) dx_1 dx_3}{\sqrt{(x_1 - \kappa_1)(x_1 - \kappa_2)(x_1 - x_3)(\kappa_2 - x_3)}}$$

and

$$D(\kappa_1, \kappa_2) := \int_0^{\kappa_2} \int_{\nu_1}^{\nu_0} \frac{(x_1 - x_2)\rho_1(x_1)\rho_2(x_2) dx_1 dx_2}{\sqrt{(x_1 - \kappa_1)(x_1 - \kappa_2)(x_2 - \kappa_1)(\kappa_2 - x_2)}}.$$

It follows from Lemma 5.4 that A , B and D are analytic functions in $(\kappa_1, \kappa_2) \in (\nu_3, 0) \times (0, \nu_1)$. Moreover, $D \neq 0$ in that domain, which implies that Ω_1 and Ω_2 are analytic in $(\kappa_1, \kappa_2) \in (\nu_3, 0) \times (0, \nu_1)$.

Denote by \mathcal{J} the Jacobian of the frequency map $(\kappa_1, \kappa_2) \mapsto (\Omega_1(\kappa_1, \kappa_2), \Omega_2(\kappa_1, \kappa_2))$,

$$\mathcal{J}(\kappa_1, \kappa_2) := \left| \frac{\partial(\Omega_1, \Omega_2)}{\partial(\kappa_1, \kappa_2)} \right| = \frac{\pi^2}{D^4} \begin{vmatrix} A_{\kappa_1} D - AD_{\kappa_1} & A_{\kappa_2} D - AD_{\kappa_2} \\ B_{\kappa_1} D - BD_{\kappa_1} & B_{\kappa_2} D - BD_{\kappa_2} \end{vmatrix} \quad (5.11)$$

Since $\mathcal{J}(\kappa_1, \kappa_2)$ is analytic in $(\kappa_1, \kappa_2) \in (\nu_3, 0) \times (0, \nu_1)$, either $\mathcal{J}(\kappa_1, \kappa_2) \neq 0$ in an open dense subset of $(\nu_3, 0) \times (0, \nu_1)$ or

$$\mathcal{J}(\kappa_1, \kappa_2) = 0 \quad \text{for any } (\kappa_1, \kappa_2) \in (\nu_3, 0) \times (0, \nu_1). \quad (5.12)$$

We are going to compute the limit of $\mathcal{J}(\kappa_1, \kappa_2)$ as $k_1 \rightarrow \nu_3 + 0$. To do this we will need the following auxiliary Lemma.

Lemma 5.5. Let $f(x, \kappa)$ be a function on $(a, b) \times (a, b)$ such that f and its partial derivatives f_x , f_κ and $f_{x\kappa}$ exist and are continuous and bounded on $(a, b) \times (a, b)$. Consider the function $F(\kappa) := \int_a^\kappa \frac{f(x, \kappa)}{\sqrt{\kappa - x}} dx$. Then

$$(a) F(\kappa) = 2f(a, \kappa)\sqrt{\kappa - a} + O(|\kappa - a|^{3/2})$$

$$(b) F'(\kappa) = \frac{f(a, \kappa)}{\sqrt{\kappa - a}} + O(\sqrt{\kappa - a})$$

where the estimates above are uniform in $\kappa \in (a, b)$.

Proof of Lemma 5.5. An integration by parts leads to

$$F(\kappa) = -\frac{1}{2} \int_a^{\kappa-0} f(x, \kappa) d\sqrt{\kappa - x} = 2f(a, \kappa)\sqrt{\kappa - a} + 2 \int_a^\kappa f_x(x, \kappa)\sqrt{\kappa - x} dx \quad (5.13)$$

that together with the boundedness of f_x proves (a). Differentiating (5.13) with respect to κ and using the boundedness of f_x , f_κ , and $f_{x\kappa}$, we prove (b). \square

The expression for $A(\kappa_1, \kappa_2)$ can be rewritten in the form

$$A(\kappa_1, \kappa_2) = \int_{\nu_3}^{\kappa_1} \frac{f(x_3, \kappa_1; \kappa_2)}{\sqrt{\kappa_1 - x_3}} dx_3$$

where

$$f(x_3, \kappa_1; \kappa_2) := \frac{\rho_3(x_3)}{\sqrt{\kappa_2 - x_3}} \int_0^{\kappa_2} \frac{(x_2 - x_3)\rho_2(x_2) dx_2}{\sqrt{(x_2 - \kappa_1)(\kappa_2 - x_2)}}.$$

For any given $\kappa_2 \in (0, \nu_1)$ the functions $f(x_3, \kappa_1; \kappa_2)$ and $\frac{\partial f(x_3, \kappa_1; \kappa_2)}{\partial \kappa_2}$ satisfy the conditions of Lemma 5.5 (with $x \equiv x_3$, $\kappa = \kappa_1$, $a = \nu_3 < 0 < b < 0$) in view of Remark 5.3. Applying the Lemma we get

$$A(\kappa_1, \kappa_2) = \left(2 \frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}} \right) \sqrt{\kappa_1 - \nu_3} + o(\sqrt{\kappa_1 - \nu_3}) \quad (5.14)$$

$$\frac{\partial A(\kappa_1, \kappa_2)}{\partial \kappa_1} = \left(\frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}} \right) / \sqrt{\kappa_1 - \nu_3} + o(1/\sqrt{\kappa_1 - \nu_3}) \quad (5.15)$$

and

$$\frac{\partial A(\kappa_1, \kappa_2)}{\partial \kappa_2} = 2 \frac{\partial}{\partial \kappa_2} \left(\frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}} \right) \sqrt{\kappa_1 - \nu_3} + o(\sqrt{\kappa_1 - \nu_3}) \quad (5.16)$$

as $\kappa_1 \rightarrow \nu_3 + 0$. In the same way one obtains

$$B(\kappa_1, \kappa_2) = \left(2 \frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa_2}} \right) \sqrt{\kappa_1 - \nu_3} + o(\sqrt{\kappa_1 - \nu_3}) \quad (5.17)$$

$$\frac{\partial B(\kappa_1, \kappa_2)}{\partial \kappa_1} = \left(\frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa_2}} \right) / \sqrt{\kappa_1 - \nu_3} + o(1/\sqrt{\kappa_1 - \nu_3}) \quad (5.18)$$

and

$$\frac{\partial B(\kappa_1, \kappa_2)}{\partial \kappa_2} = 2 \frac{\partial}{\partial \kappa_2} \left(\frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa_2}} \right) \sqrt{\kappa_1 - \nu_3} + o(\sqrt{\kappa_1 - \nu_3}) \quad (5.19)$$

as $\kappa_1 \rightarrow \nu_3 + 0$. Note also that for any $\kappa_2 \in (0, \nu_1)$, $D(\kappa_1, \kappa_2)$ is a continuous (even real-analytic) function with respect to κ_1 on the whole interval $(-\infty, 0)$.

Consider the limit $\delta(\kappa_2) := \lim_{\kappa_1 \rightarrow \nu_3 + 0} \pi^{-2} D^3 \mathcal{J}(\kappa_1, \kappa_2)$ for $\kappa_2 \in (0, \nu_1)$. It follows from (5.11) and (5.14)-(5.19) that

$$\begin{aligned} \pi^{-2} D^3 \mathcal{J} &= (A_{\kappa_1} B_{\kappa_2} - B_{\kappa_1} A_{\kappa_2}) + \frac{D_{\kappa_1}}{D} (-AB_{\kappa_2} + A_{\kappa_2} B) + \frac{D_{\kappa_2}}{D} (AB_{\kappa_1} - A_{\kappa_1} B) \\ &= (A_{\kappa_1} B_{\kappa_2} - B_{\kappa_1} A_{\kappa_2}) + o(1) \\ &= 2 \left(\frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \right)^2 \left(\int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}} \right)^2 \frac{\partial}{\partial \kappa_2} \left(\frac{\int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa_2}}}{\int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}}} \right) + o(1) \end{aligned}$$

as $\kappa_1 \rightarrow \nu_3 + 0$. Hence,

$$\delta(\kappa_2) = 2 \left(\frac{\rho_3(\nu_3)}{\sqrt{\kappa_2 - \nu_3}} \right)^2 \left(\int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}} \right)^2 \frac{\partial}{\partial \kappa_2} \left(\frac{\int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa_2}}}{\int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}}} \right). \quad (5.20)$$

Suppose that (5.12) holds. Then $\delta(\kappa_2) = 0$ for any $\kappa_2 \in (0, \nu_1)$ and it follows from (5.20) that there is a constant $C \neq 0$ such that

$$\int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa_2}} = C \int_0^{\kappa_2} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa_2 - x_2}} \quad (5.21)$$

for any $\kappa_2 \in (0, \nu_1)$.

Lemma 5.6. *Let (X, \tilde{g}) be a Liouville billiard table of classical type. Then the geodesic flow of the restriction $\tilde{l} = \tilde{g}|_{\Gamma}$ of the Riemannian metric \tilde{g} to the boundary Γ is completely integrable. A functionally independent with \tilde{l} integral of the geodesic flow of \tilde{l} is given by the restriction $\tilde{I} = \tilde{I}_2|_{\Gamma}$ of \tilde{I}_2 to Γ and the level set $\{\tilde{l} = 1, \tilde{I} = \kappa\}$ is non empty if and only if $\kappa \in [0, \nu_0]$. In action-angle coordinates the rotation function corresponding to the Liouville torus $\tilde{T}_{\kappa} := \{\tilde{l} = 1, \tilde{I} = \kappa\}$ ⁶ for $\kappa \in (0, \nu_1)$ is*

$$\rho(\kappa) = 2 \int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa}} / \int_0^{\kappa} \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa - x_2}}. \quad (5.22)$$

Proof of Lemma 5.6. It follows from the construction of the Liouville billiard tables that the mapping $\sigma|_{\mathbb{T}_N^2} : \mathbb{T}_N^2 \rightarrow \Gamma$ is a double branched covering of the boundary Γ , where

$$\mathbb{T}_N^2 = \{(\theta_1 \pmod{\omega_1}, \theta_2 \pmod{\omega_2}, \theta_3 = N)\} \subset C.$$

⁶This set has two connected components that correspond to two Liouville tori with the same rotation function.

In the coordinates $\{(\theta_1, \theta_2)\}$ on \mathbb{T}_N^2 we get the following expressions for the metric $l = (\sigma|_{\mathbb{T}_N^2})^*\tilde{l}$ and the integral $I = (\sigma|_{\mathbb{T}_N^2})^*\tilde{I}$

$$\begin{aligned} dl^2 &= (\varphi_1 - \varphi_2) \left((\varphi_1 - \nu_3) d\theta_1^2 + (\varphi_2 - \nu_3) d\theta_2^2 \right), \\ dI^2 &= (\varphi_1 - \varphi_2) \left(\varphi_2(\varphi_1 - \nu_3) d\theta_1^2 + \varphi_1(\varphi_2 - \nu_3) d\theta_2^2 \right). \end{aligned}$$

Applying the Legendre transformation corresponding to l we obtain the following system of equations for the level set $T_\kappa := \{l = 1, I = \kappa\}$,

$$\begin{aligned} L &= \frac{1}{\varphi_1 - \varphi_2} \left(\frac{p_1^2}{\varphi_1 - \nu_3} + \frac{p_2^2}{\varphi_2 - \nu_3} \right) = 1 \\ I &= \frac{1}{\varphi_1 - \varphi_2} \left(\varphi_2 \frac{p_1^2}{\varphi_1 - \nu_3} + \varphi_1 \frac{p_2^2}{\varphi_2 - \nu_3} \right) = \kappa \end{aligned}$$

that leads to the following expression of the impulses on T_κ ,

$$p_1(\theta_1)^2 = (\varphi_1(\theta_1) - \nu_3)(\varphi_1(\theta_1) - \kappa) \geq 0 \quad (5.23)$$

$$p_2(\theta_2)^2 = (\varphi_2(\theta_2) - \nu_3)(\kappa - \varphi_2(\theta_2)) \geq 0. \quad (5.24)$$

In particular, $T_\kappa \neq \emptyset$ if and only if $\kappa \in [0, \nu_0]$. Hence, the projection of T_κ into the base \mathbb{T}_N^2 is given by the union of the sets

$$A'_\kappa := \{(\theta_1, \theta_2) : 0 \leq \theta_1 \leq \omega_1, -f_2(\kappa) \leq \theta_2 \leq f_2(\kappa)\}$$

and

$$A''_\kappa := \{(\theta_1, \theta_2) : 0 \leq \theta_1 \leq \omega_1, -f_2(\kappa) + \omega_2/2 \leq \theta_2 \leq f_2(\kappa) + \omega_2/2\}.$$

As the sets A'_κ and A''_κ have the same image under the projection $\sigma|_{\mathbb{T}_N^2} : \mathbb{T}_N^2 \rightarrow \Gamma$ we restrict our attention only to the set A'_κ . It follows from (5.23)-(5.24) that the mapping $r_+ : A'_\kappa \rightarrow T^*\mathbb{T}_N^2$,

$$(\theta_1, \theta_2) \mapsto (\theta_1, \theta_2; \sqrt{(\varphi_1(\theta_1) - \nu_3)(\varphi_1(\theta_1) - \kappa)}, \pm \sqrt{(\varphi_2(\theta_2) - \nu_3)(\kappa - \varphi_2(\theta_2))}),$$

parametrizes one of the two connected components of the set $\tilde{T}_\kappa = \{\tilde{l} = 1, \tilde{I} = \kappa\} \subset T^*X$. By Liouville-Arnold formula we get the following formulas for the corresponding actions

$$\begin{aligned} J_1(\kappa) &= \frac{2}{\pi} \int_0^{\omega_1} \sqrt{(\varphi_1(\theta_1) - \nu_3)(\varphi_1(\theta_1) - \kappa)} d\theta_1 \\ &= \frac{8}{\pi} \int_{\nu_1}^{\nu_0} \sqrt{(x_1 - \nu_3)(x_1 - \kappa)} \rho_1(x_1) dx_1 \end{aligned} \quad (5.25)$$

$$\begin{aligned} J_2(\kappa) &= \frac{2}{\pi} \int_{-f_2(\kappa)}^{f_2(\kappa)} \sqrt{(\varphi_2(\theta_2) - \nu_3)(\kappa - \varphi_2(\theta_2))} d\theta_2 \\ &= \frac{4}{\pi} \int_0^\kappa \sqrt{(x_2 - \nu_3)(\kappa - x_2)} \rho_2(x_2) dx_2 \end{aligned} \quad (5.26)$$

In the corresponding action-angle coordinates the Hamiltonian L becomes $L = L^0(J_1, J_2)$, where L^0 is smooth, and the frequency vector ω of the invariant torus $\{J_1 = c_1, J_2 = c_2\}$ is $\omega = -\left(\frac{\partial L^0}{\partial J_1}(c_1, c_2), \frac{\partial L^0}{\partial J_2}(c_1, c_2)\right)$. Then, differentiating the relation

$$L^0(J_1(\kappa), J_2(\kappa)) \equiv 1$$

with respect to $\kappa \in (0, \nu_1)$ we get (5.22). \square

We need the following technical Lemma.

Lemma 5.7. *Let $m < 0 < M$ be real constants, $F_1 \in C^1([0, M])$, and $F_2 \in C^1([m, 0])$. Then*

$$\int_0^M \frac{F_1(\sqrt{t})}{\sqrt{t}\sqrt{t-\alpha}} dt = -2F_1(0) \log \sqrt{-\alpha} + O(1) \quad (5.27)$$

and

$$\int_m^\alpha \frac{F_2(\sqrt{-t})}{\sqrt{-t}\sqrt{\alpha-t}} dt = 2F_2(0) \log \sqrt{-\alpha} + O(1) \quad (5.28)$$

as $\alpha \rightarrow 0 - 0$.

Proof. We have

$$\int_0^M \frac{F_1(\sqrt{t})}{\sqrt{t}\sqrt{t-\alpha}} dt = 2F_1(0) \int_0^{\sqrt{M}} \frac{1}{\sqrt{u^2 - \alpha}} du + O(1) = -2F_1(0) \log \sqrt{-\alpha} + O(1).$$

The proof of (5.28) is similar and we omit it. \square

Lemma 5.7 can be applied to the two integrals in (5.22) using Remark 5.3. In this way we obtain

$$\int_{\nu_1}^{\nu_0} \frac{\sqrt{x_1 - \nu_3} \rho_1(x_1) dx_1}{\sqrt{x_1 - \kappa}} = -2G_1^+(0) \sqrt{\nu_1 - \nu_3} \log \sqrt{\nu_1 - \kappa} + O(1)$$

and

$$\int_0^\kappa \frac{\sqrt{x_2 - \nu_3} \rho_2(x_2) dx_2}{\sqrt{\kappa - x_2}} = 2G_2^-(0) \sqrt{\nu_1 - \nu_3} \log \sqrt{\nu_1 - \kappa} + O(1).$$

On the other hand, Remark 5.3 and assumption (A_3) , (2), in Sect. 3.1 imply

$$G_2^-(0) = -\sqrt{-2\varphi_2''(\omega_2/4)^{-1}} = -\sqrt{2\varphi_1''(0)^{-1}} = -G_1^+(0),$$

and by (5.22) we obtain

$$\rho(\kappa) \rightarrow 2 \text{ as } \kappa \rightarrow \nu_1 - 0.$$

As by (5.21), $\rho \equiv \text{const}$ we conclude that $\rho \equiv 2$ on the interval $(0, \nu_1)$. The latter implies that all the geodesics of Γ lying on a torus \tilde{T}_κ with $\kappa \in (0, \nu_1)$ (see Lemma 5.6) are periodic. Using the analyticity of the billiard table and considering the Poincaré map in a tubular neighborhood of the ‘‘hyperbolic’’ level set $\{l = 1, I = \nu_1\}$ we obtain that any geodesics of Γ corresponding to some $\kappa \in [\nu_1, \nu_2]$ is periodic as well. As the level sets $\{l = 1, I = 0\}$ and $\{l = 1, I = \nu_0\}$ consists of periodic geodesics we see that all the geodesics on Γ are periodic. Hence, the assumption that the Jacobian \mathcal{J} of the frequency map vanishes in an open subset of $(\kappa_1, \kappa_2) \in (\nu_3, 0) \times (0, \nu_1)$ implies that all the geodesics of Γ are periodic. The case (B) can be studied by the same argument. \square

6 Proof of Theorem 1 and Theorem 3

In this section we prove Theorem 3 and Theorem 1 formulated in the introduction. Let (X, g) be a 3-dimensional analytic Liouville billiard table of classical type such that $\Gamma := \partial X$ admits at least one non closed geodesic.

We will prove a more general result than Theorem 3 which requires only finite smoothness of K_t . Namely, fix $d > 1/2$ and $\ell > 4[2d] + 11$, where $[2d]$ is the entire part of $2d$ and d is the exponent in (H_1) . Denote by $C^\ell(\Gamma, \mathbb{R})$ the corresponding class of Hölder continuous functions.

Theorem 6.1. *Let $[0, 1] \ni t \mapsto K_t$ be a continuous curve in $C^\ell(\Gamma, \mathbb{R})$ and suppose that it satisfies (H_1) and (H_2) , where ℓ and d are fixed as above. If K_0 and K_1 are invariant with respect to the group of symmetries G , then $K_0 \equiv K_1$.*

Proof. Given $\alpha > 0$ and $\tau > 2$ we denote by Ω_α^τ the set of all frequencies $(\Omega_1, \Omega_2) \in \mathbb{R}^2$ satisfying the Diophantine condition

$$\text{For any } (k_1, k_2, k_3) \in \mathbb{Z}^3, (k_1, k_2) \neq (0, 0) : \quad |\Omega_1 k_1 + \Omega_2 k_2 + k_3| \geq \frac{\alpha}{(|k_1| + |k_2|)^\tau}.$$

Note that the set $\Omega^\tau := \cup_{\alpha>0} \Omega_\alpha^\tau$ is of full Lebesgue measure in \mathbb{R}^2 for any $\tau > 2$ fixed (cf. [8, Proposition 9.9]). Then it follows from Theorem 5.1 that the subset of $U_1 \cup U_2$ filled by invariant tori Λ with frequencies in Ω^τ is dense in $U_1 \cup U_2$. Take $0 < \tau - 2 \ll 1$ so that $\ell > ([2d] + 1)(\tau + 2) + 7$. Then we apply [12, Theorem 1.1] for any Λ in that family. By Remark 5.2 we have

$$\mathcal{R}_{K_0, \mu}(\Lambda) = \mathcal{R}_{K_t, \mu}(\Lambda) \tag{6.1}$$

for any $t \in [0, 1]$ and for any torus Λ with frequency in Ω^τ , where $\mu = \langle \pi^+(\xi), n_g \rangle^{-1}$. By continuity we obtain (6.1) for any Liouville torus Λ lying in the part $U_1 \cup U_2$ of $B^*\Gamma$ corresponding to the boundary cases. Finally, Theorem 6.1 follows from (6.1) and Theorem 4.1. \square

Proof of Theorem 1. Let (X, g) be a 3-dimensional analytic Liouville billiard table of classical type and let $\mu = 1$ or $\mu = \langle \pi^+(\xi), n_g \rangle^{-1}$. Assume that $K \in C(\Gamma, \mathbb{R})$ is invariant with respect to the group of symmetries $G = (\mathbb{Z}/2\mathbb{Z})^3$ of Γ and let the mean value of $\mu \cdot K$ on any periodic orbit of the billiard ball map be zero. It follows from Theorem 5.1 that the set filled by Liouville tori Λ of the billiard ball map with frequency vectors $\Omega := (\Omega_1, \Omega_2) \in \mathbb{Q} \times \mathbb{Q}$ is dense in the part of $B^*\Gamma$ corresponding to boundary cases. Let Λ be such a *rational* torus. In action-angle coordinates, $\Lambda \cong \mathbb{R}^2/\mathbb{Z}^2$. There exists $N \in \mathbb{N}$ and two relatively prime numbers $p, q \in \mathbb{Z}$ such that $\Omega \equiv \left(\frac{p}{N}, \frac{q}{N} \right) \pmod{\mathbb{Z}^2}$. Hence, there is an affine change of coordinates on $\mathbb{R}^2/\mathbb{Z}^2$ such that $\Omega \equiv (1/N, 0) \pmod{\mathbb{Z}^2}$. Denote,

$$\mathcal{D} := \{(x, y) : 0 \leq x < 1/N, 0 \leq y \leq 1\}.$$

Using the invariance of Λ and of the Leray form on Λ with respect to B^N we obtain,

$$\int_{\Lambda} (\mu \cdot K) \lambda = \frac{1}{N} \sum_{k=1}^N \int_{\mathcal{D}} (B^*)^k (\mu \cdot K) \lambda = \int_{\mathcal{D}} \left(\frac{1}{N} \sum_{k=1}^N (B^*)^k (\mu \cdot K) \right) \lambda = 0 \tag{6.2}$$

as by assumption the mean $\sum_{k=1}^N (B^*)^k (\mu \cdot K)$ vanishes. Using the density of rational tori Λ in boundary cases, equality (6.2), and Theorem 4.1 we see that $K \equiv 0$. \square

7 Appendix: Frequencies of integrable billiard tables

In this appendix we collect the necessary facts used for the computation of the frequency map in Sect. 5. Our main task is to derive formula (7.9) for the frequencies of the billiard ball map.

Let (X, g) , $n = \dim X \geq 2$, be a billiard table with non-empty locally convex boundary Γ . Consider the reflection map at the boundary,

$$\rho : TX|_{\Gamma} \rightarrow TX|_{\Gamma}, \quad \xi \mapsto \xi - 2g(\xi, n_g)n_g, \quad (7.1)$$

where $TX|_{\Gamma} := \{\xi \in TX : \pi(\xi) \in \Gamma\}$ is the restriction of the tangent bundle to Γ , $\pi : TX \rightarrow X$ is the natural projection onto the base, and n_g is the inward unit normal to the boundary. The restriction ρ is an involution on $TX|_{\Gamma}$ the set of fixed point of which coincides with $T\Gamma \subseteq TX|_{\Gamma}$. Note that ρ preserves the values of the Hamiltonian $H_g(\xi) := \frac{1}{2}g(\xi, \xi)$ and when restricted to the unit spherical bundle $S_g|_{\Gamma} := \{\xi \in TX|_{\Gamma} : \|\xi\|_g = 1\}$ it coincides with the mapping $r : \Sigma \rightarrow \Sigma$ considered in Sect. 2 if we identify vectors and covectors with the help of the Legendre transform,

$$FL_g : TX \rightarrow T^*X, \quad \xi \mapsto g(\xi, \cdot).$$

More generally, the notions and mappings considered in Sect. 2 have their analogs on TX via the Legendre transform.

Denote by α_g the Liouville 1-form on TX given by $\alpha_g(v)(\cdot) := g(v, d_v\pi(\cdot))$ where $v \in TX$ and (\cdot) stands for an arbitrary element of $T_v(TX)$. Note that the differential $\omega_g := d\alpha_g$ of the 1-form α_g corresponds to the symplectic form $dp \wedge dx$ on the cotangent bundle T^*X via the Legendre transform.

Lemma 7.1. *The reflection map $\rho : TX|_{\Gamma} \rightarrow TX|_{\Gamma}$ satisfies the following properties:*

- (a) *the reflection ρ preserves the restriction of the Liouville form α_g to $TX|_{\Gamma}$;*
- (b) *the reflection ρ preserves the values of the Hamiltonian $H_g(\xi) = \frac{1}{2}g(\xi, \xi)$;*
- (c) *in the case when (X, \tilde{g}) is a Liouville billiard table the reflection ρ corresponding to the Riemannian metric \tilde{g} preserves the values of the pairwise commuting integrals \tilde{I}_k ($k = 1, 2$) of the billiard flow (cf. Proposition 3.3).*

Proof of Lemma 7.1. (a) Let $t \mapsto v(t)$ be a smooth curve in $TX|_{\Gamma}$ defined in an open neighborhood of $t = 0$ such that $v(0) = v \in T_x X$, $x \in \Gamma$, and $\dot{v}(0) = \Xi \in T_v(TX|_{\Gamma})$. One has

$$\begin{aligned} \alpha_g(\rho(v))(d_v\rho(\Xi)) &= g(\rho(v), d_{\rho(v)}\pi \circ d_v\rho(\Xi)) = g\left(\rho(v), \frac{d}{dt}\pi(\rho(v(t))|_{t=0})\right) \\ &= g\left(v - 2g(v, n_g)n_g, \frac{d}{dt}\pi(v(t))|_{t=0}\right) = g\left(v, \frac{d}{dt}\pi(v(t))|_{t=0}\right) \\ &= \alpha_g(v)(\Xi) \end{aligned} \quad (7.2)$$

where we have used that $\pi \circ \rho = \pi$ and that $\frac{d}{dt}\pi(v(t))|_{t=0} \in T_x\Gamma$ is orthogonal to n_g . This proves statement (a). The proof of (b) is straightforward and we omit it. Statement (c) was established in the proof of Proposition 3.3. \square

Now we will describe a special variant of the symplectic gluing procedure introduced by Lazutkin in [8, § 4]. The main idea is to identify parts of the boundary $\partial(TX)$ of the configuration

space TX of the billiard flow in order to “eliminate” the reflections and obtain a new “glued” configuration space together with a smooth billiard flow on it. Note that the glued configurations space becomes a smooth symplectic manifold so that the billiard flow is a smooth Hamiltonian system on it. Divide the boundary of TX into three parts

$$\partial(TX) = TX|_{\Gamma} = T^-X|_{\Gamma} \sqcup T^+X|_{\Gamma} \sqcup T\Gamma$$

where $T^{\pm}X|_{\Gamma} := \{\xi \in TX_{\Gamma} : \pm g(\xi, n_g) > 0\}$ and $T\Gamma$ is assumed naturally embedded into $TX|_{\Gamma}$. Note that

$$\rho|_{T^-X} : T^-X|_{\Gamma} \rightarrow T^+X|_{\Gamma} \quad (7.3)$$

is a diffeomorphism and the elements of $T\Gamma \subseteq TX|_{\Gamma}$ are fixed points of ρ . Now, using (7.3) we identify the points $\xi^- \in T^-X$ and $\rho(\xi^-) \in T^+X$ of the boundary of $TX \setminus T\Gamma$ and obtain a new *glued* space \widetilde{TX}^{ρ} that we supply with the factor topology so that the projection $\pi_{\rho} : TX \setminus T\Gamma \rightarrow \widetilde{TX}^{\rho}$,

$$TX \setminus T\Gamma \ni \xi \mapsto \begin{cases} \xi & \text{if } \xi \notin \partial(TX) \\ \{\xi, \rho(\xi)\} & \text{if } \xi \in T^-X|_{\Gamma} \\ \{\rho^{-1}(\xi), \xi\} & \text{if } \xi \in T^+X|_{\Gamma} \end{cases},$$

is continuous.

Definition 7.2. *The billiard flow of (X, g) is called completely integrable if there exist n functionally independent integrals $Q_1, \dots, Q_n \equiv H_g \in C^{\infty}(TX, \mathbb{R})$ of the billiard flow such that $\forall 1 \leq k, l \leq n$, $\{Q_k, Q_l\} = 0$, and $\forall 1 \leq k \leq n \ \forall \xi \in TX|_{\Gamma}$, $Q_k(\rho(\xi)) = Q_k(\xi)$.*

Assume that the billiard flow on TX is *completely integrable*. Denote by X_g the Hamiltonian vector field on TX with Hamiltonian H_g . The following Proposition follows from Lemma 7.1 (a), (b), and is a special case of the symplectic gluing developed in [8, § 4].

Proposition 7.3. *There exists a smooth differentiable structure on \widetilde{TX}^{ρ} , a symplectic form $\tilde{\omega}_g$ on \widetilde{TX}^{ρ} , and functions $\tilde{Q}_k \in C^{\infty}(\widetilde{TX}^{\rho}, \mathbb{R})$ ($1 \leq k \leq n$), such that the projection*

$$\pi_{\rho} : TX \setminus T\Gamma \rightarrow \widetilde{TX}^{\rho} \quad (7.4)$$

is smooth, $\pi_{\rho}^(\tilde{\omega}_g) = \omega_g$, and $\pi_{\rho}^*(\tilde{Q}_k) = Q_k$ for any $1 \leq k \leq n$. In particular, the Hamiltonian vector field \tilde{X}_g corresponding to $\tilde{H}_g := \tilde{Q}_n$ is completely integrable in \widetilde{TX}^{ρ} and $(\pi_{\rho})_*(X_g) = \tilde{X}_g$.*

Denote,

$$\mathcal{T} := \pi_{\rho}(T^{\pm}X|_{\Gamma}) \subset \widetilde{TX}^{\rho}.$$

Note that \mathcal{T} is a disjoint union of connected non-intersecting embedded hypersurfaces in \widetilde{TX}^{ρ} that are transversal to the Hamiltonian vector field \tilde{X}_g .

Let $c = (c_1, \dots, c_n)$ be a regular value of the “momentum” map

$$M : \widetilde{TX}^{\rho} \rightarrow \mathbb{R}^n, \quad \xi \mapsto (\tilde{Q}_1(\xi), \dots, \tilde{Q}_n(\xi))$$

and let \tilde{T}_c be a connected component of the level set $M^{-1}(c)$. The compactness of X implies that \tilde{T}_c is compact. By the Liouville-Arnold theorem \tilde{T}_c is diffeomorphic to the n dimensional

torus \mathbb{T}^n and one can introduce *action-angle* coordinates in a tubular neighborhood of \tilde{T}_c in \widetilde{TX}^ρ (see [2]). Assume that

$$\tilde{T}_c \cap \mathcal{T} \neq \emptyset$$

is a non-empty compact set. In this case we will call \tilde{T}_c *glued Liouville torus*. As \tilde{X}_g is tangent to \tilde{T}_c and transversal to \mathcal{T} the submanifolds \tilde{T}_c and \mathcal{T} intersect transversally. Hence, $\tilde{T}_c \cap \mathcal{T}$ is a disjoint union of finitely many compact embedded submanifolds in \tilde{T}_c . Denote by $m \geq 1$ the number of the connected components of $\tilde{T}_c \cap \mathcal{T}$. The proof of the following Lemma is straightforward and we omit it.

Lemma 7.4.

- (a) *The connected components of $\tilde{T}_c \cap \mathcal{T}$ are diffeomorphic to \mathbb{T}^{n-1} ;*
- (b) *The closure of any of the connected components of $\tilde{T}_c \setminus \mathcal{T}$ is diffeomorphic to $[0, 1] \times \mathbb{T}^{n-1}$. The n -torus \tilde{T}_c is obtained by a “cyclic” gluing together of all $m \geq 1$ copies of $[0, 1] \times \mathbb{T}^{n-1}$ along their boundaries;*
- (c) *Let \mathcal{S}_c be a connected component of $\tilde{T}_c \cap \mathcal{T}$ and let $\Lambda_c = p_+(\pi_\rho^{-1}(\mathcal{S}_c))$ where $p_+ : TX|_\Gamma \rightarrow T\Gamma$ denotes the orthogonal projection $\xi \mapsto \xi - g(n_g, \xi) n_g$ onto $T\Gamma$. Then the number $m \geq 1$ of the connected components of $\tilde{T}_c \cap \mathcal{T}$ is the minimal power of the billiard ball map B that leaves Λ_c invariant, i.e., $B^m(\Lambda_c) = \Lambda_c$.*⁷

Choose a component \mathcal{S}_c of $\tilde{T}_c \cap \mathcal{T}$ and a basis of cycles $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{n-1}$ of its homology group as well as a *transversal* cycle $\tilde{\gamma}_n$ in \tilde{T}_c so that $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ is a basis of the homology group of \tilde{T}_c . Let $\{\tilde{J}_1, \dots, \tilde{J}_n; \tilde{\theta}_1(\text{mod } 2\pi), \dots, \tilde{\theta}_n(\text{mod } 2\pi)\}$ be action-angle coordinates in a tubular neighborhood of the glued Liouville torus \tilde{T}_c that corresponds to the cycles $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$, i.e., $\forall 1 \leq k \leq n$,

$$\tilde{J}_k = \frac{1}{2\pi} \int_{\tilde{\gamma}_k} \tilde{\alpha}_g,$$

where $\tilde{\alpha}_g := (\pi_\rho)_*(\alpha_g)$ is the push-forward of the Liouville form α_g onto \widetilde{TX}^ρ . In the action-angle coordinates,

$$\tilde{X}_g = \eta_1(\tilde{J}) \frac{\partial}{\partial \tilde{\theta}_1} + \dots + \eta_n(\tilde{J}) \frac{\partial}{\partial \tilde{\theta}_n},$$

where

$$\eta_k(\tilde{J}_1, \dots, \tilde{J}_n) := \frac{\partial \tilde{\mathcal{H}}_g}{\partial \tilde{J}_k}(\tilde{J}_1, \dots, \tilde{J}_n), \quad 1 \leq k \leq n. \quad (7.5)$$

It follows from the choice of the cycles $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ that \mathcal{S}_c is a section of the bundle,

$$\mathbb{T}^{n-1} \times \mathbb{T} \rightarrow \mathbb{T}^{n-1}, \quad (\tilde{\theta}_1, \dots, \tilde{\theta}_n) \mapsto (\tilde{\theta}_1, \dots, \tilde{\theta}_{n-1}).$$

As \mathcal{S}_c is transversal to \tilde{X}_g one concludes that $\eta_n(\tilde{J}) \neq 0$. It follows from our construction that the billiard ball map B^m is conjugated to the following diffeomorphism of the $n-1$ -dimensional torus $\{(\tilde{\theta}_1(\text{mod } 2\pi), \dots, \tilde{\theta}_{n-1}(\text{mod } 2\pi))\}$,

$$\tilde{\theta}_k \mapsto \tilde{\theta}_k + 2\pi \frac{\eta_k(\tilde{J})}{\eta_n(\tilde{J})}, \quad 1 \leq k \leq n-1.$$

⁷Note that we identify vectors and covectors via the Riemannian metric g .

Parameterizing the glued Liouville tori with fixed energy $\{\tilde{H}_g = 1\}$ by the values of the integrals $\tilde{Q} = (\tilde{Q}_1, \dots, \tilde{Q}_{n-1})$ we obtain the following mapping for the frequencies of B^m ,

$$(\tilde{Q}_1, \dots, \tilde{Q}_{n-1}) \mapsto (\Omega_k(\tilde{Q}_1, \dots, \tilde{Q}_{n-1}))_{1 \leq k \leq n-1} := 2\pi \left(\frac{\eta_k(J(\tilde{Q}_1, \dots, \tilde{Q}_{n-1}, 1))}{\eta_n(J(\tilde{Q}_1, \dots, \tilde{Q}_{n-1}, 1))} \right)_{1 \leq k \leq n-1} \quad (7.6)$$

where $\eta_k(\tilde{J}_1, \dots, \tilde{J}_n)$ is defined by (7.5). Finally, by partial differentiation of the identity,

$$\tilde{\mathcal{H}}_g(\tilde{J}(\tilde{Q}_1, \dots, \tilde{Q}_{n-1}, 1)) \equiv 1,$$

one gets that the frequency vector $\Omega := (\Omega_1, \dots, \Omega_{n-1})^T$ satisfies the linear relation

$$\mathbf{A}\Omega = -2\pi \mathbf{b} \quad (7.7)$$

where $\mathbf{A}(Q) := \left(\frac{\partial J_l}{\partial Q_k}(Q, 1) \right)_{1 \leq k, l \leq n-1}$, $\mathbf{b}(Q) := (\frac{\partial J_n}{\partial Q_1}(Q, 1), \dots, \frac{\partial J_n}{\partial Q_{n-1}}(Q, 1))^T$, and $\forall 1 \leq k \leq n$,

$$J_k := \frac{1}{2\pi} \int_{\gamma_k} \alpha_g, \quad (7.8)$$

where γ_k is the connected component of $\pi_\rho^{-1}(\tilde{\gamma}_k)$ lying in $T^+X|_\Gamma$ and $Q := (Q_1, \dots, Q_{n-1})$. The functions J_k ($1 \leq k \leq n$) will be called *generalized* actions of the billiard flow. Using that $\eta_n \neq 0$ one can prove that $\mathbf{A}(Q)$ is non-degenerate. Hence,

$$\Omega(Q) = -2\pi \mathbf{A}(Q)^{-1} \mathbf{b}(Q), \quad (7.9)$$

where Q_1, \dots, Q_{n-1} are the integrals of the billiard flow in a tubular neighborhood of the invariant set $\pi_\rho^{-1}(\tilde{T}_c)$ of the billiard flow.

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